Looking for intuition behind discrete topologies

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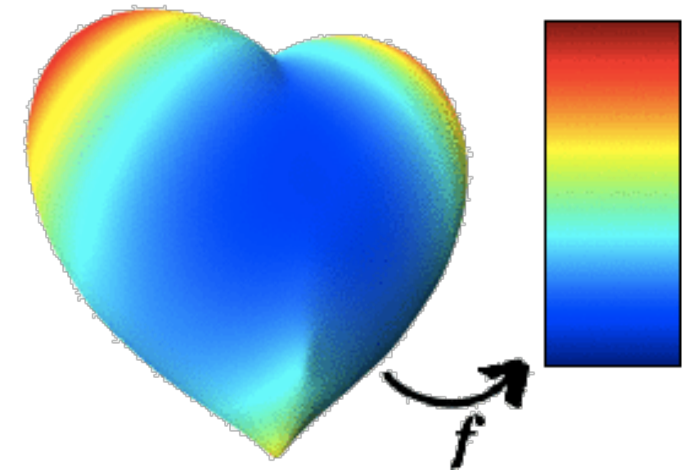
Topology-Based Methods in Data Analysis and Visualization:
TopoInVis 2011: ETH, Zürich.

matemática
puc-rio

Acknowledgements



TopoInVis organizers!!

students and colleagues:

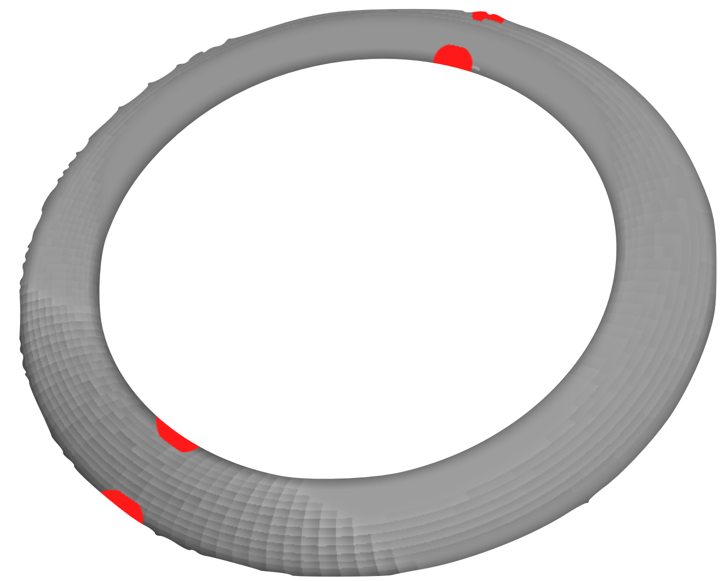
João Paixão, Renata Nascimento,
Andrei Sharf, Daniel Cohen-Or, Arik Shamir,
David Cohen-Steiner, Hélio Lopes, Geovan Tavares...

authors of the many inspirational works!

financing institute:

CNPq, FAPERJ, CAPES, EDX, IMC, Matheon...

Expectations from topology



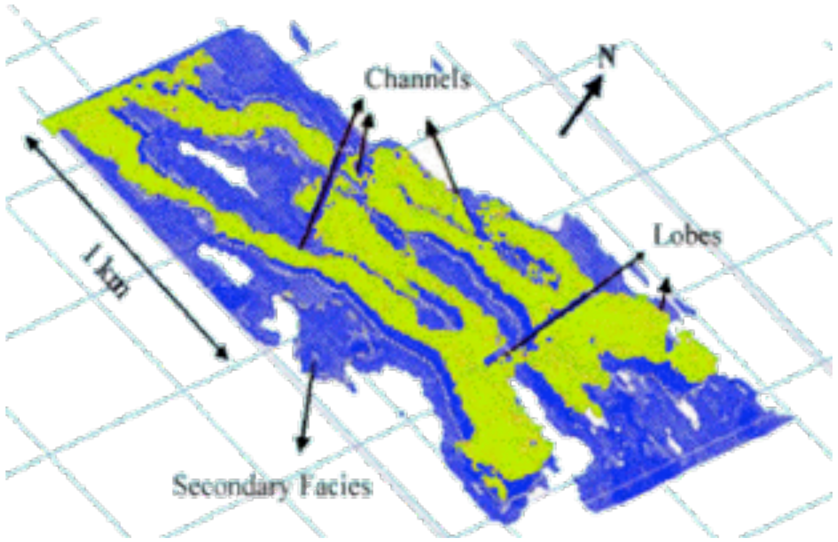
get the big picture

partially self-validated

$$\chi = \sum_{i=0}^{d} (-1)^i \cdot m_i$$

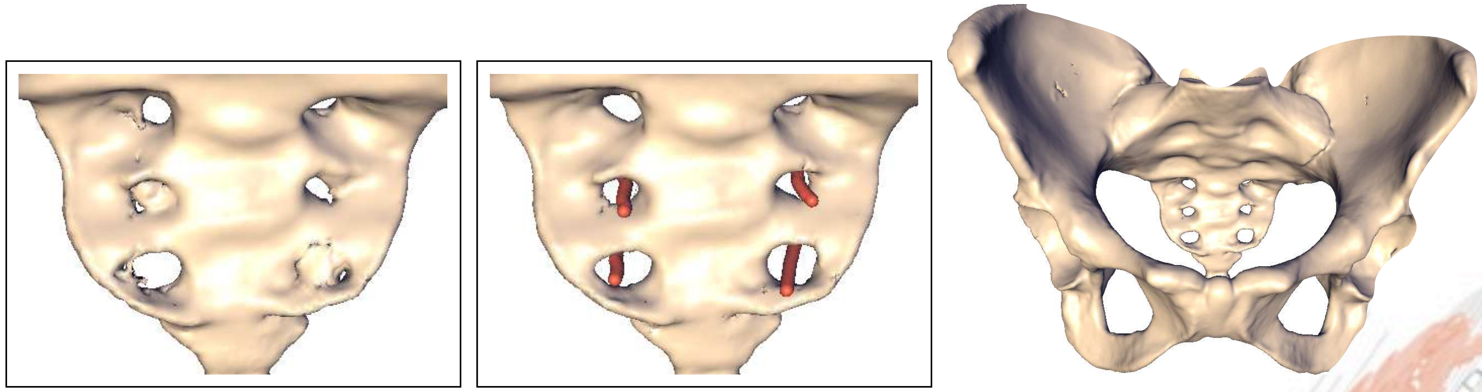
global (high info) from local (low cost)

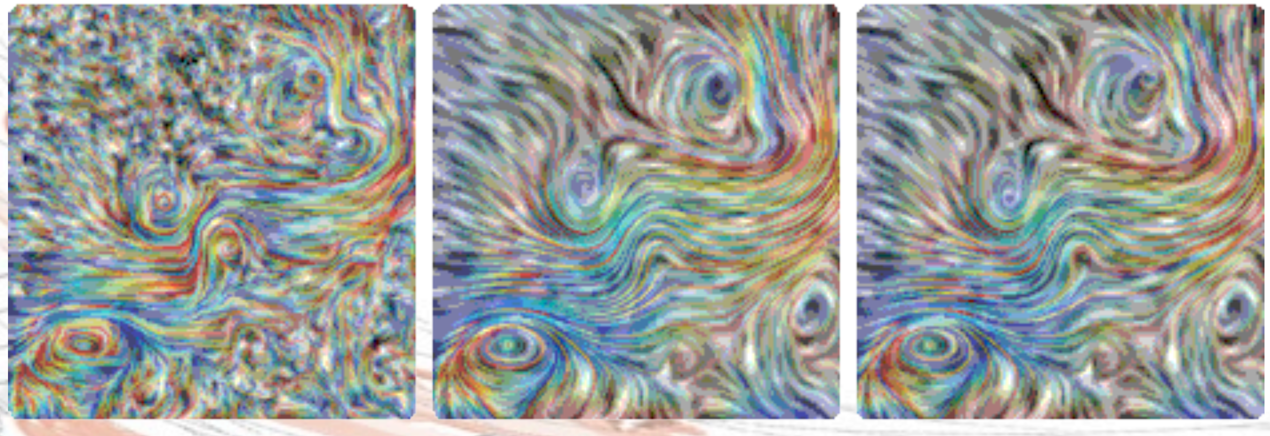
Applications that motivated me



reservoir characterization
from huge seismic data

surface extraction
and reconstruction





vector field de-noising

Intuition?

quick and ready insight
immediate apprehension or cognition

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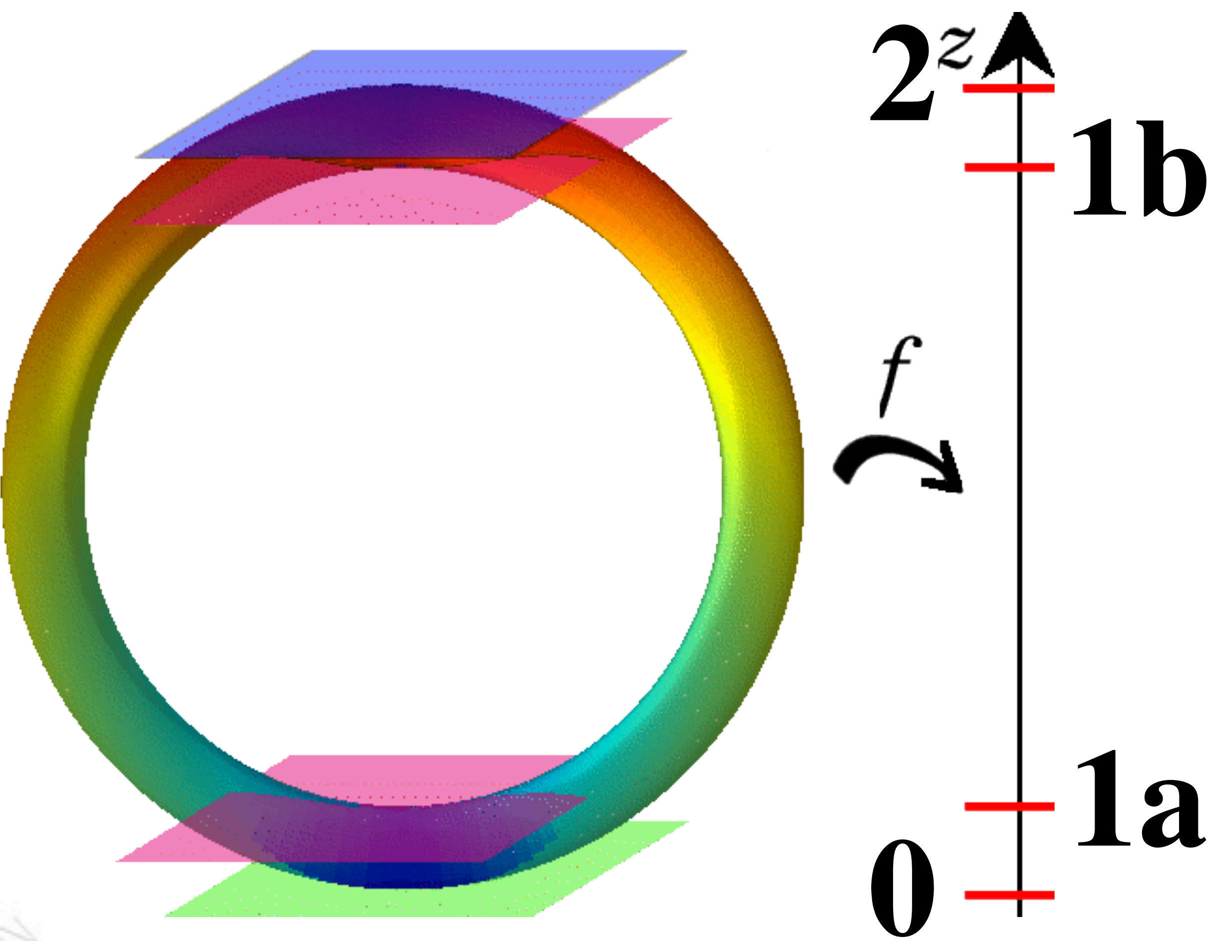
© quipu

something an industry engineer accepts

Today's filtration



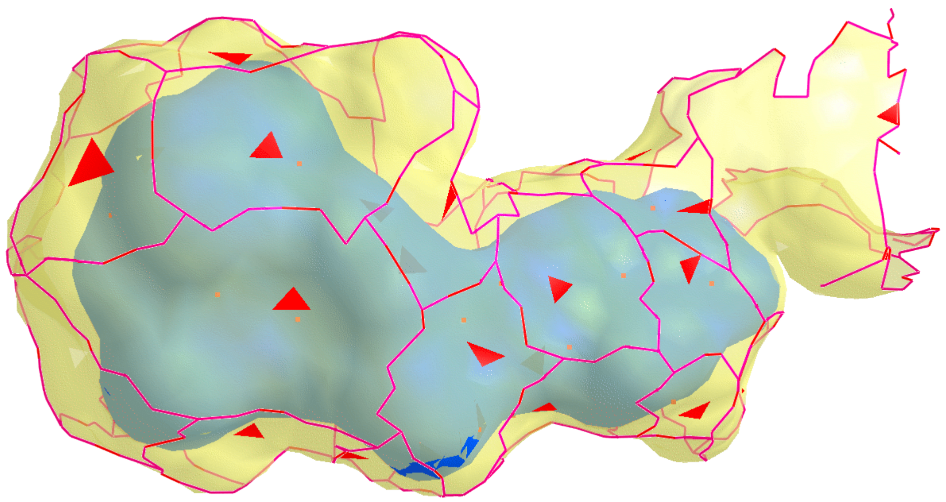
topological objects

discrete theories

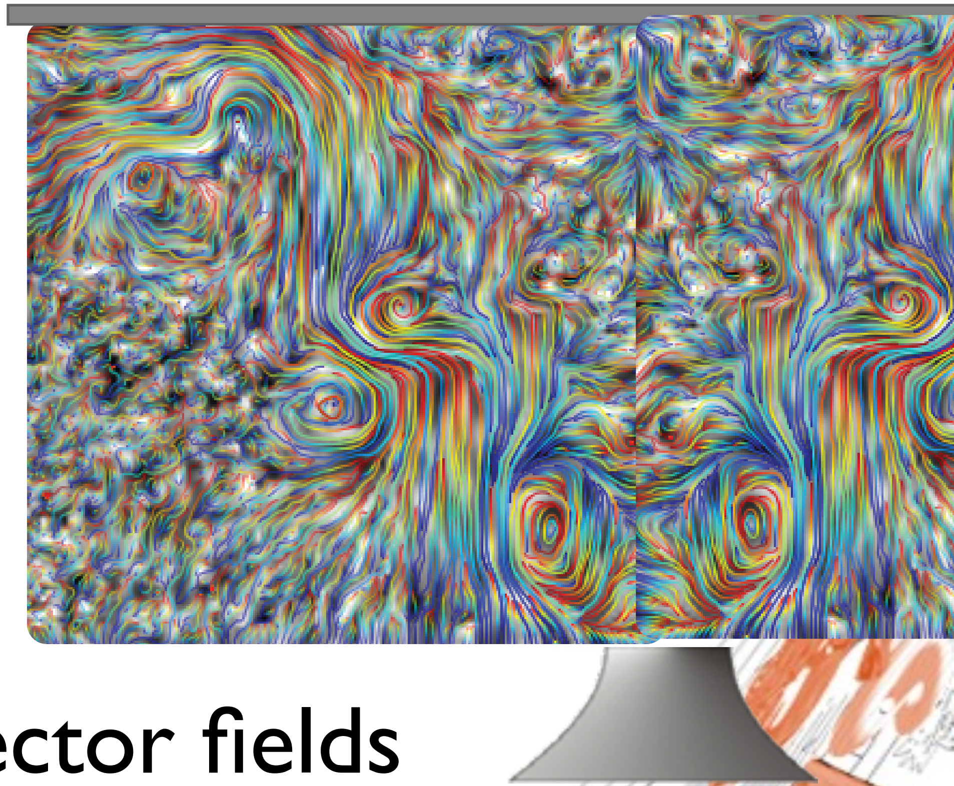
some examples

discussions

Topological objects

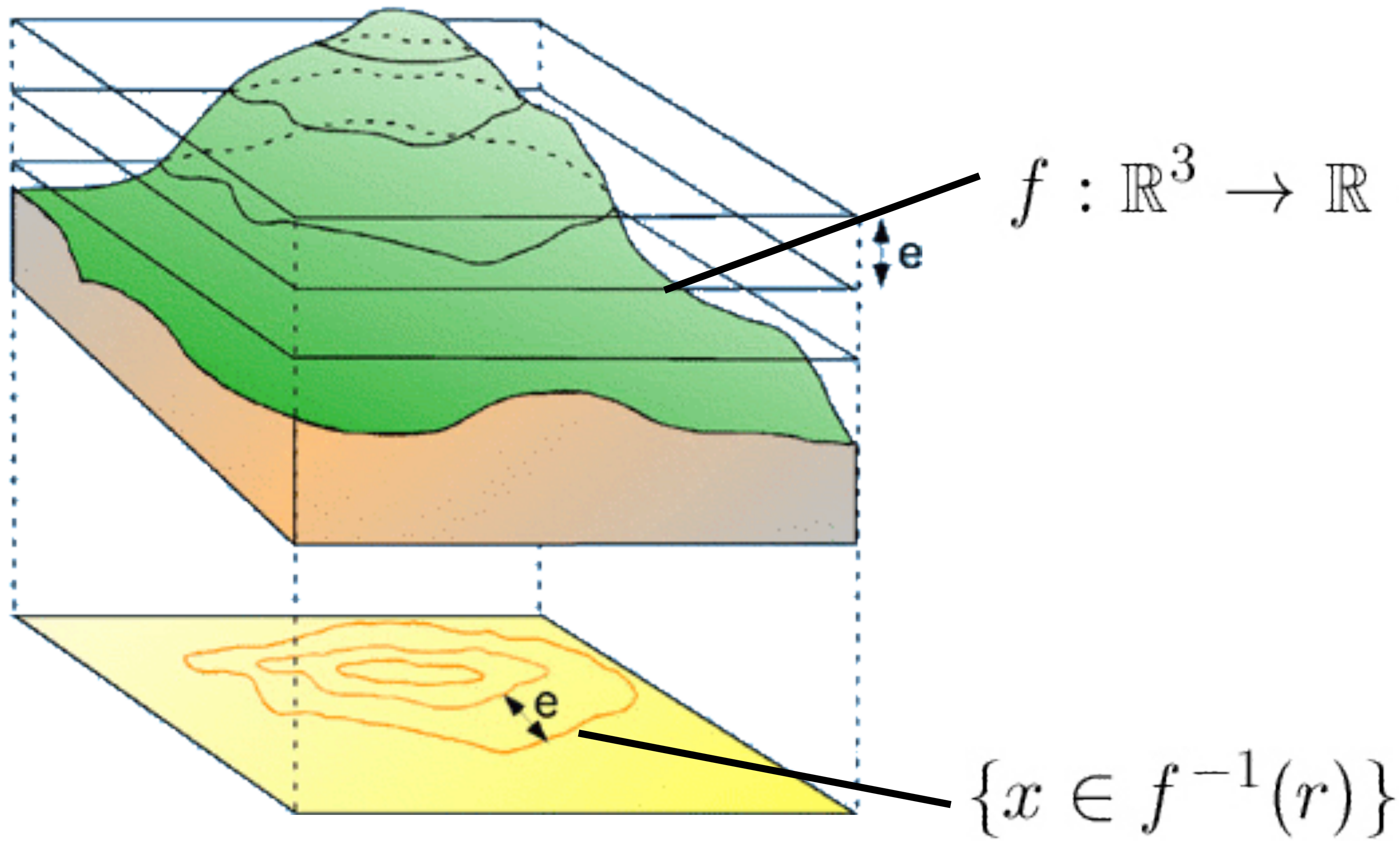


Manifolds,
Subsets of $\mathbb{R}^n$



Vector fields

Submersion intuition

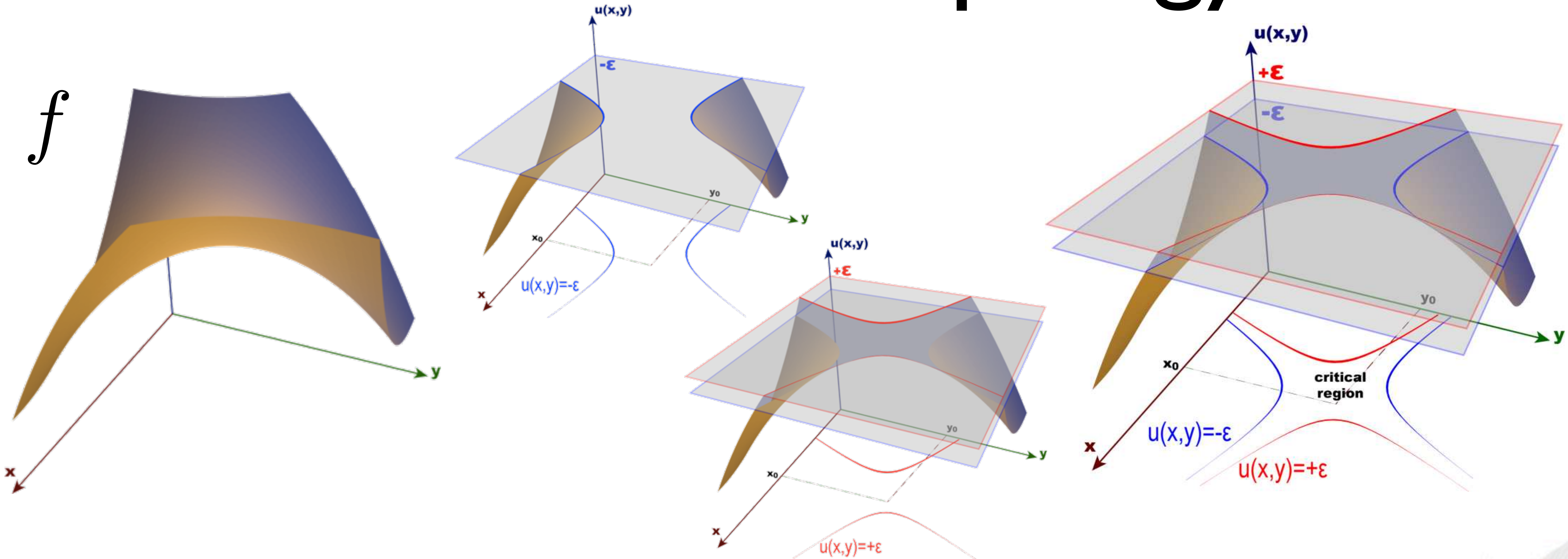


subset of $\mathbb{R}^n$ respecting a condition f

$\Rrightarrow$ closer to real data

Submersion topology

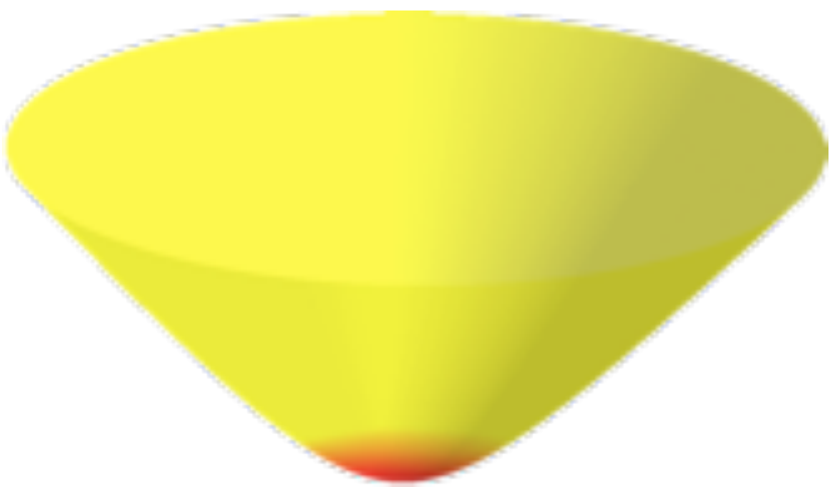
f



critical set of f (Morse lemma)

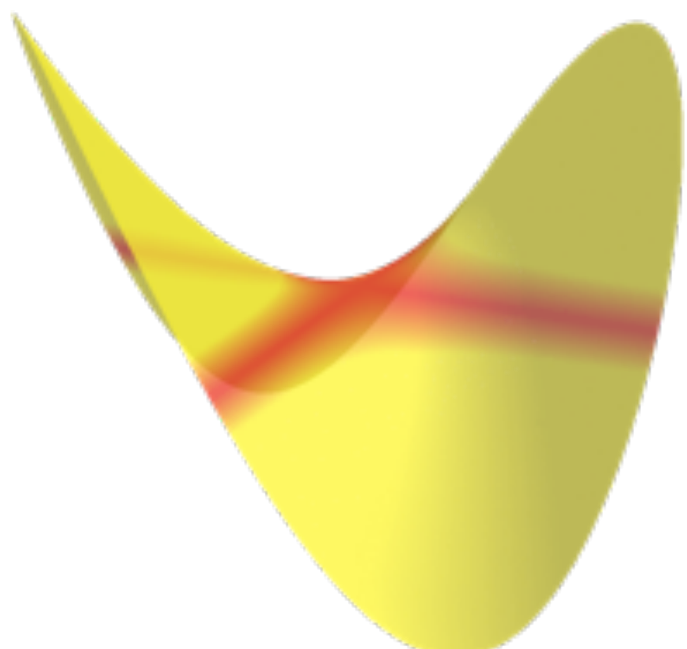
$\Rrightarrow$ global from local function analysis

Usual critical sets



minima

new component



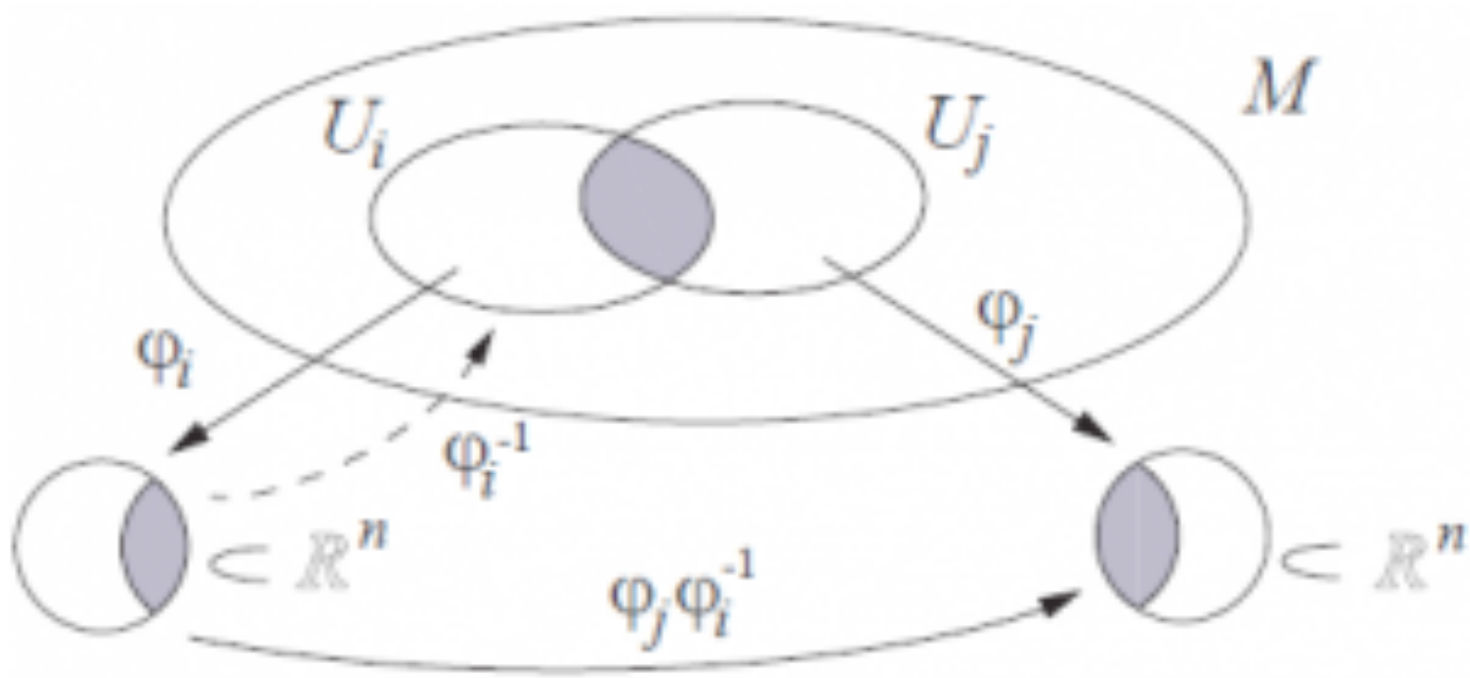
saddles

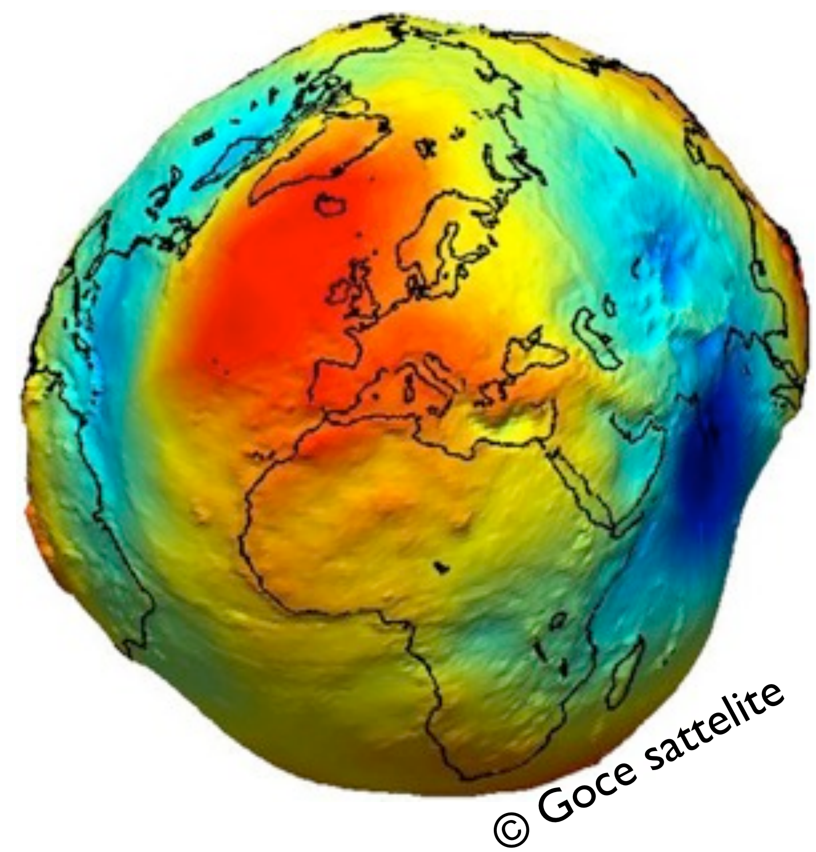
joins / splits components

maxima

end component

Immersion intuition

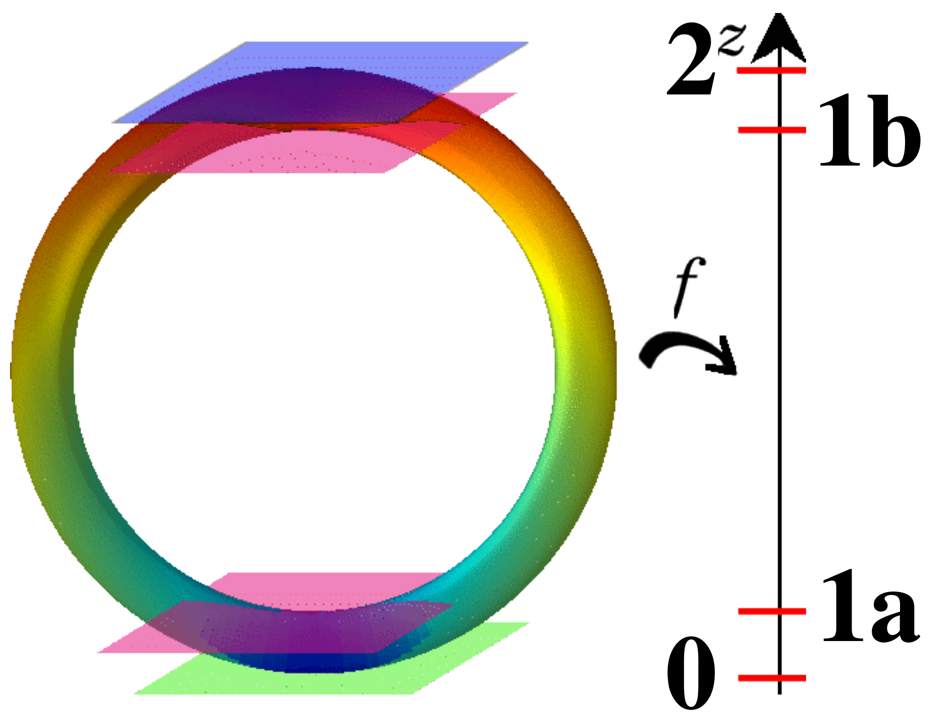


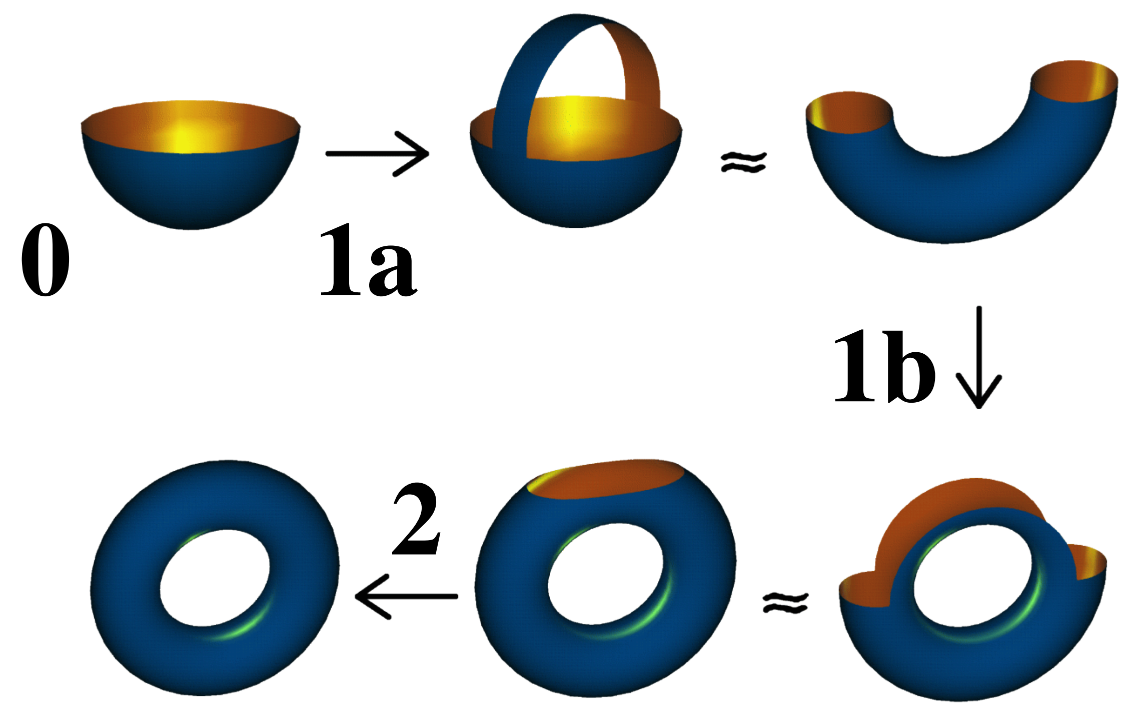


locally equivalent to $\mathbb{R}^d$

$\Rrightarrow$ intuitive differential tools

Immersion Morse topology

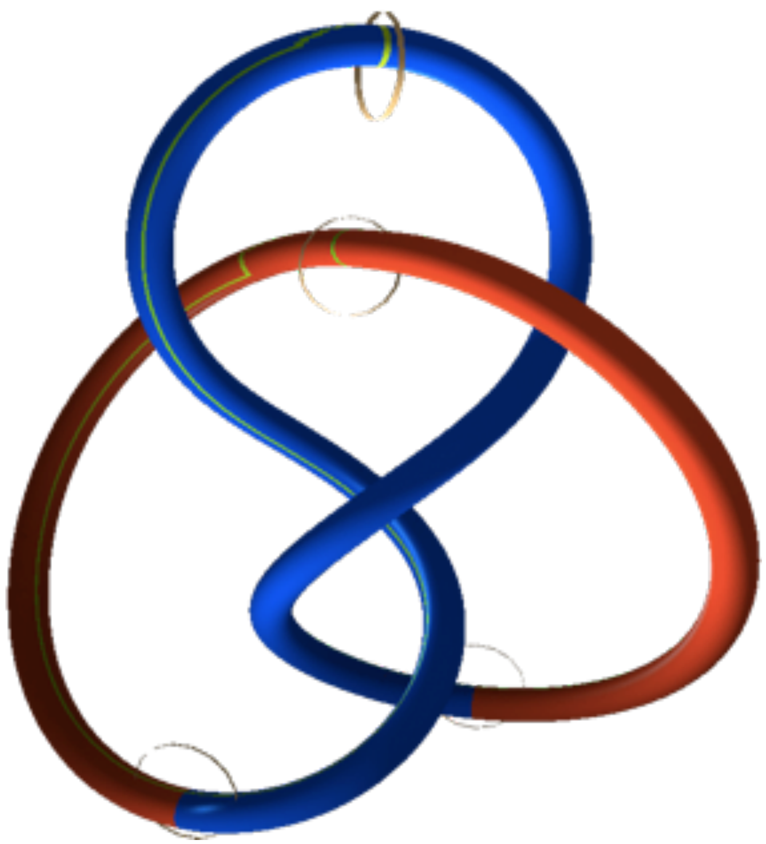


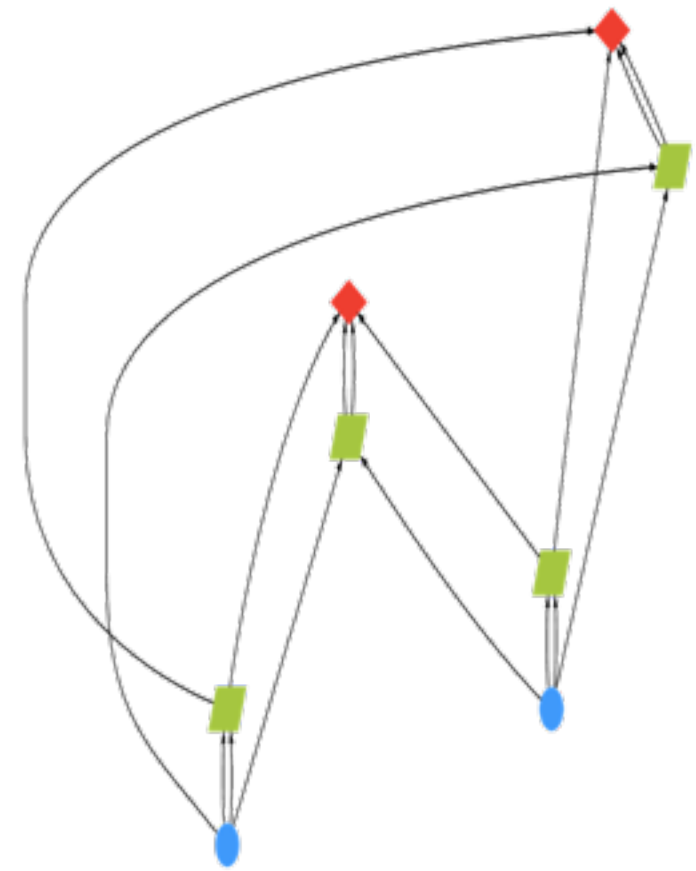


critical set of a function on the manifold

⇛ global from local function analysis

Morse-Smale complex

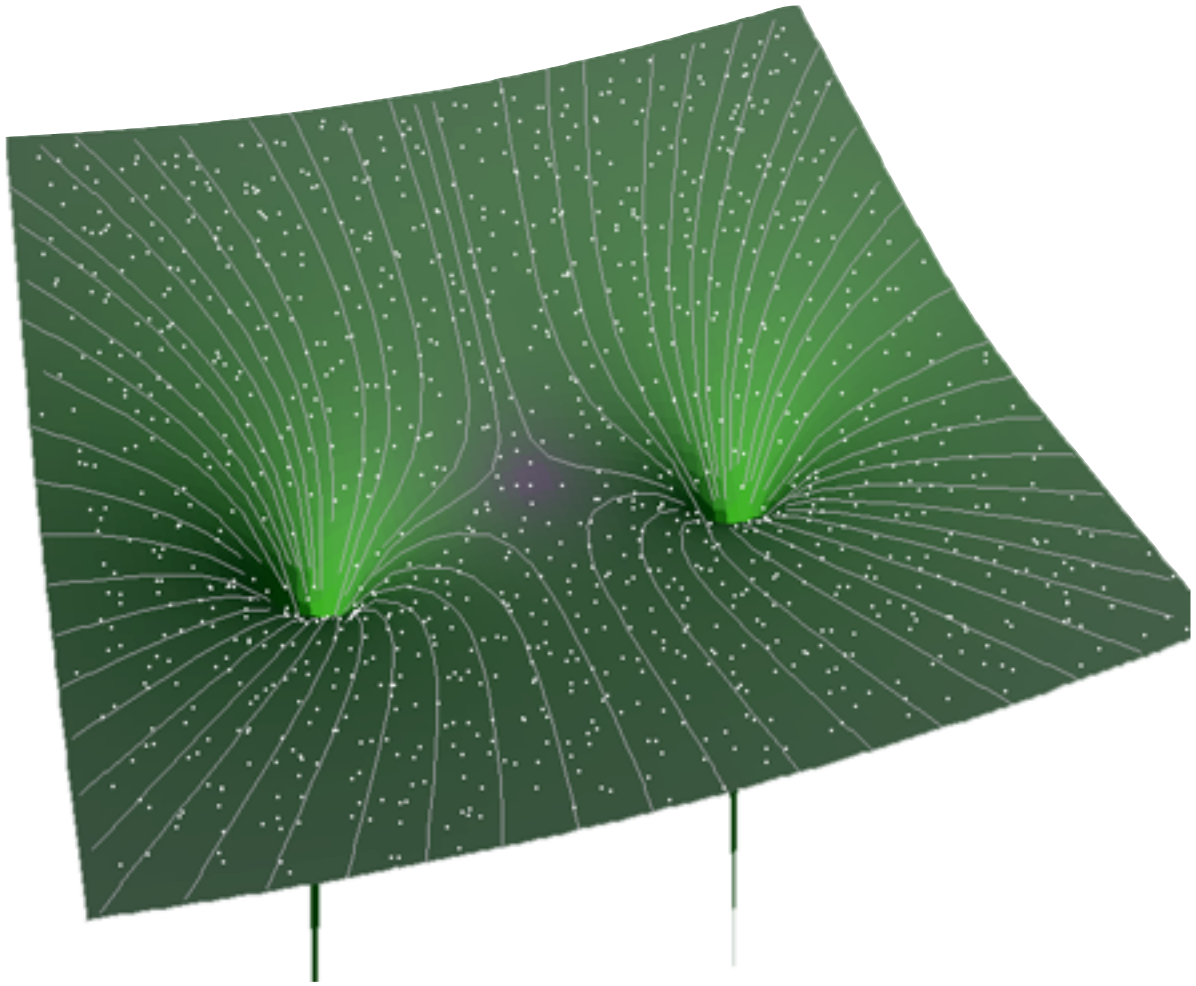


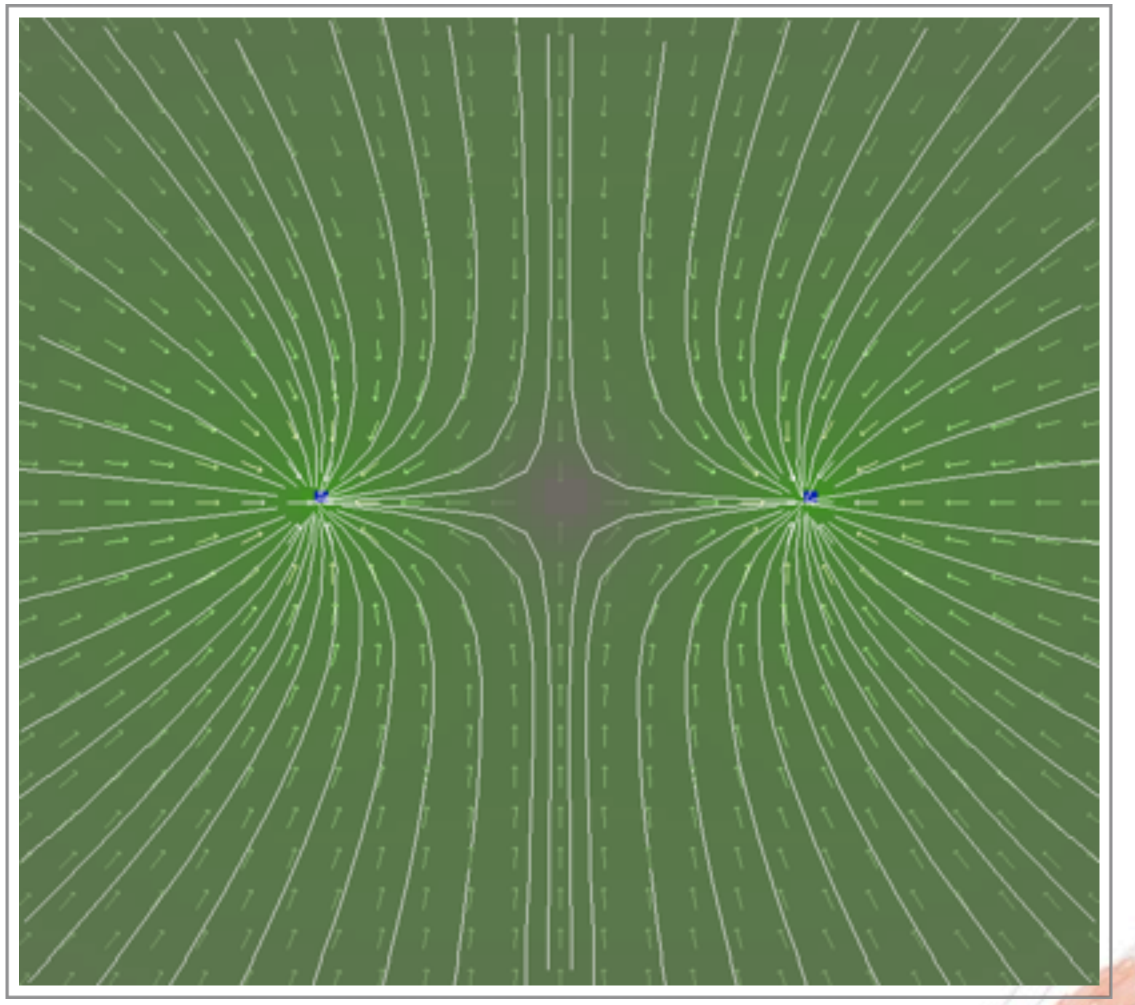


relation between critical points

$\Rrightarrow$ local function analysis + graph

Vector field

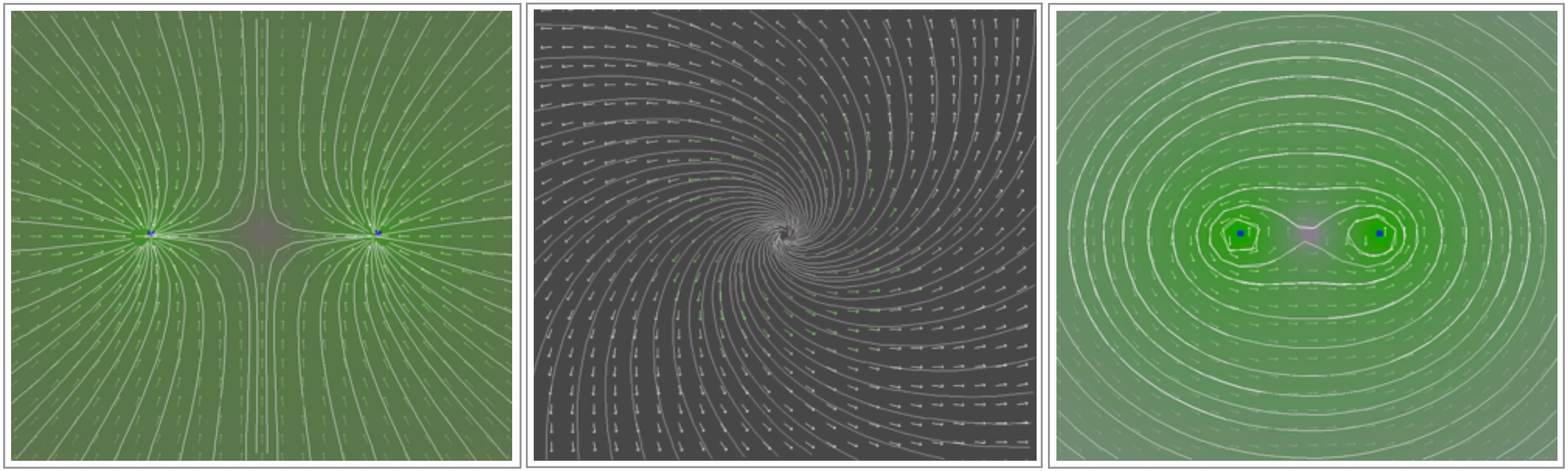




essentially tubular flow

sparse invariant sets

Vector field topology



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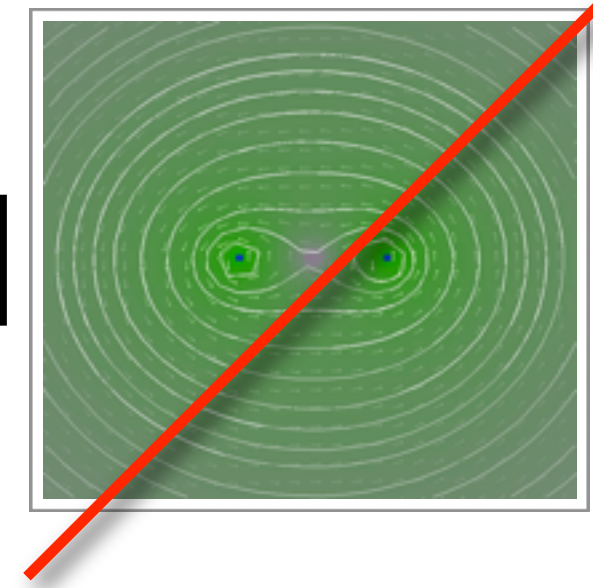
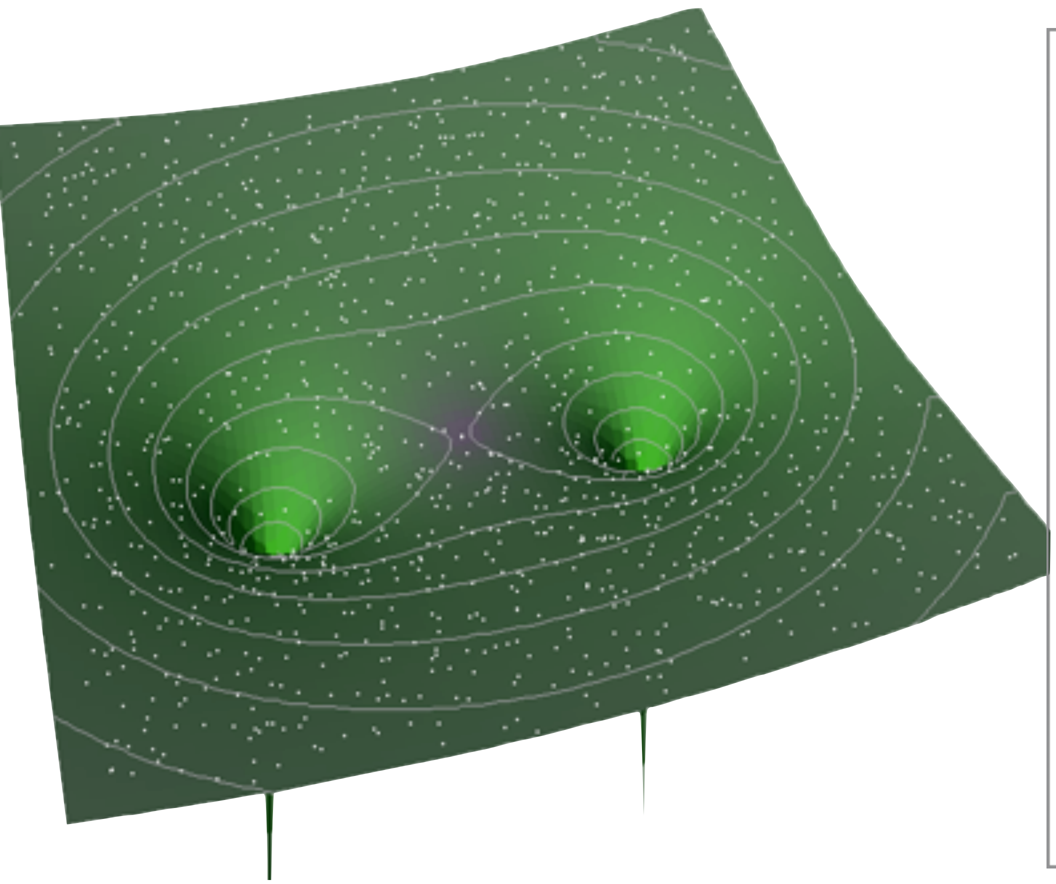
isolated singularities behavior

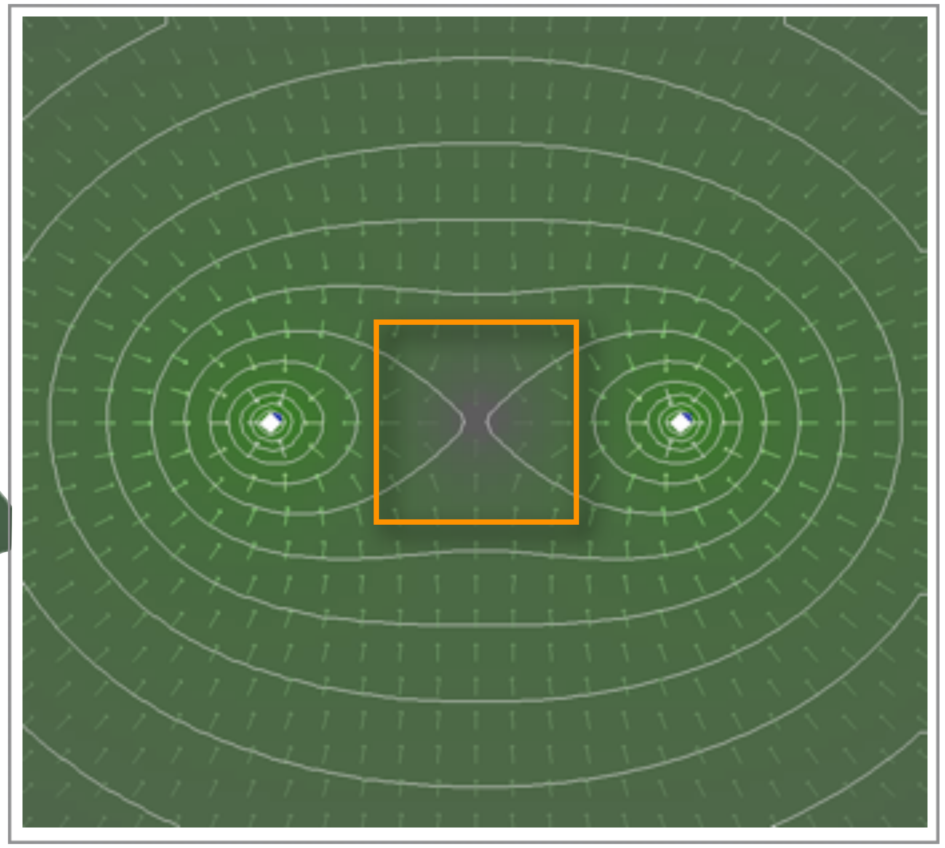
⇛ local analysis (Hartman Grobman)+graph

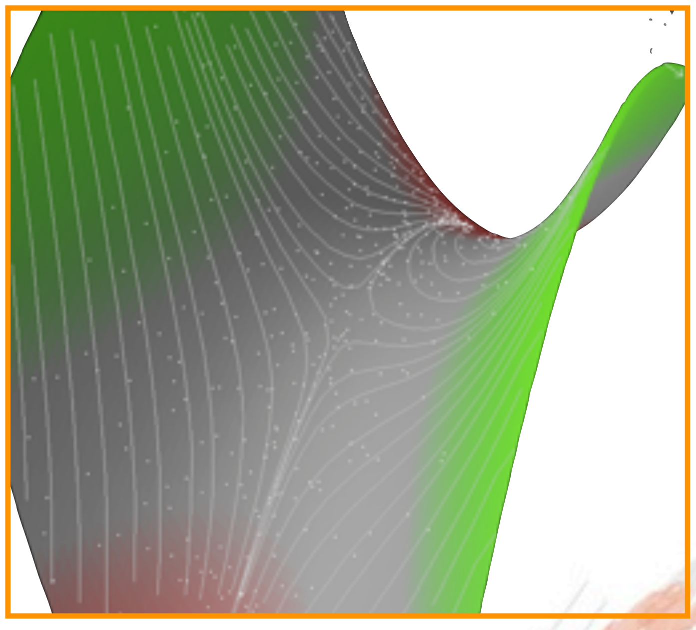
+ closed orbits + non-generic

Gradient vector field





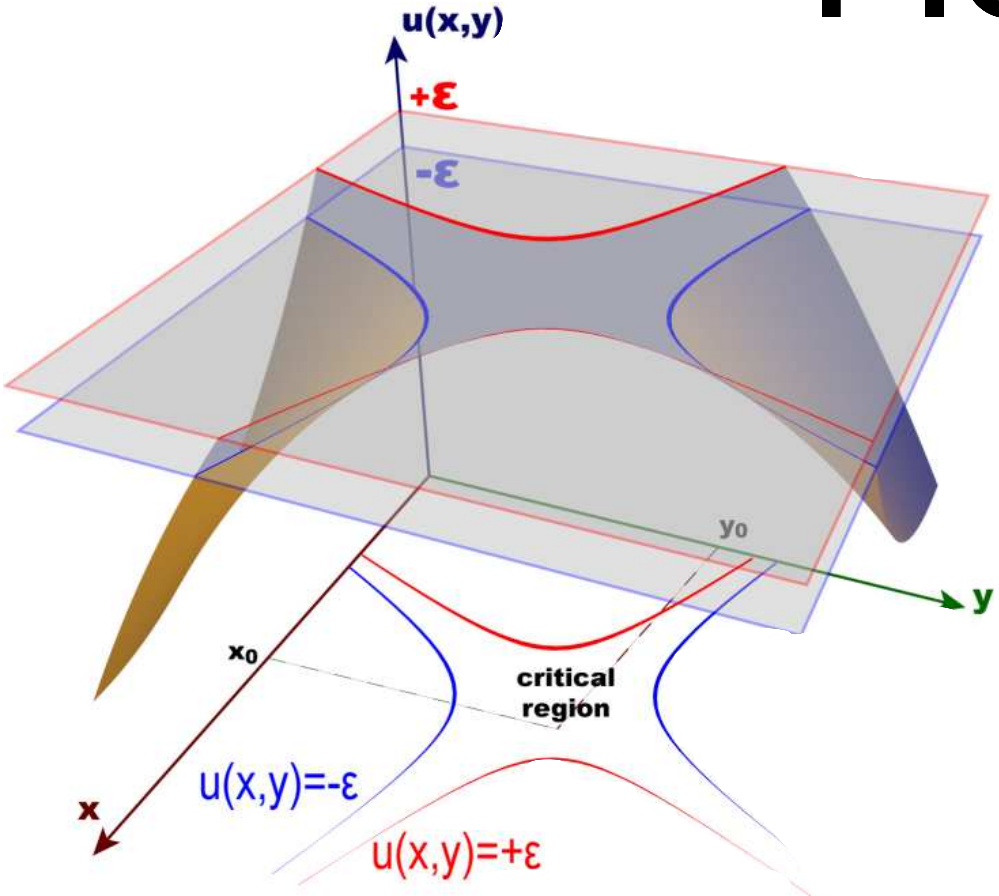




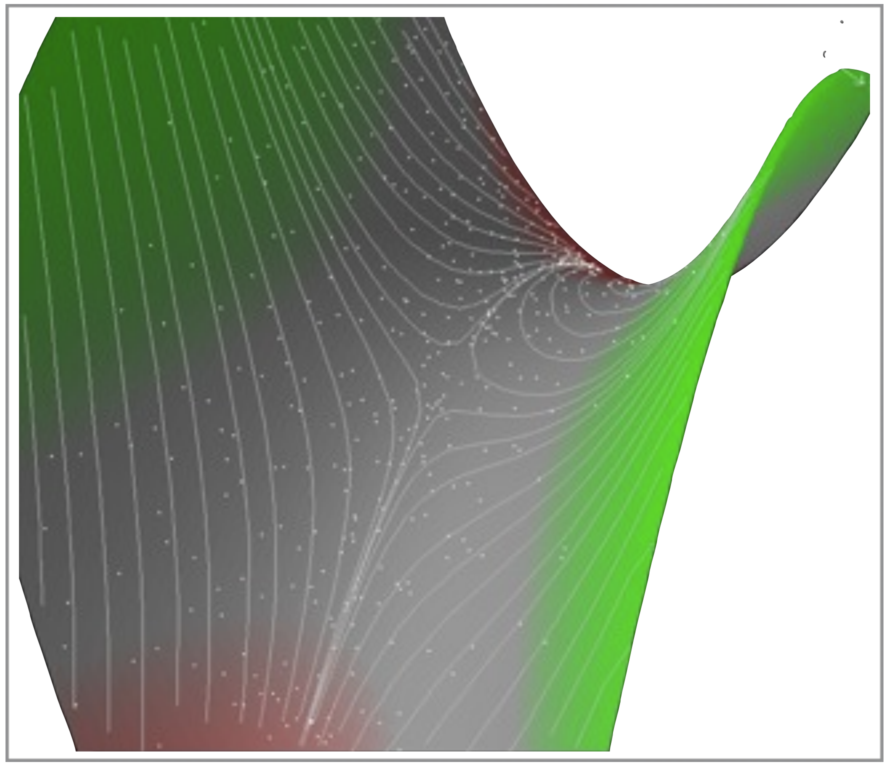
generic gradient

⇛ Morse-Smale structure

Morse theory

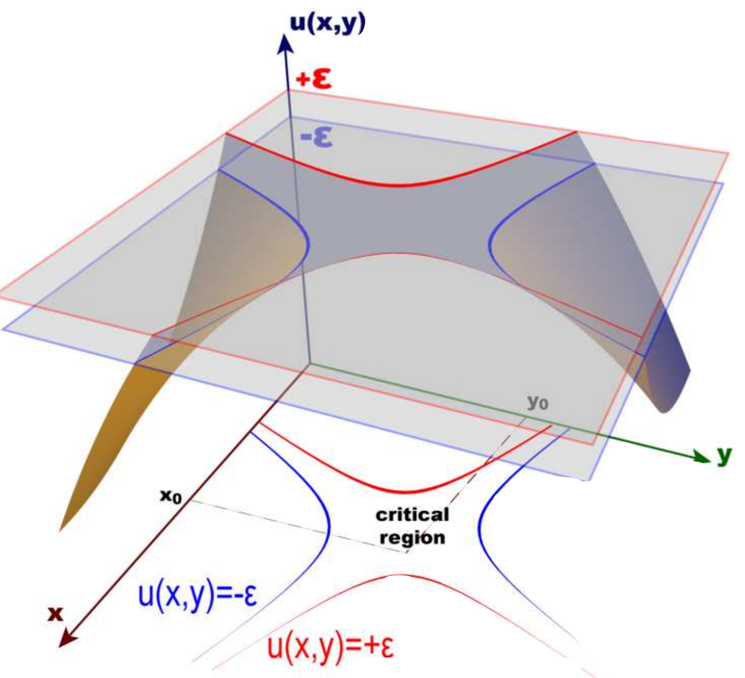


$$\chi = \sum_{i=0}^{d} (-1)^i \cdot m_i$$



topology from local function analysis

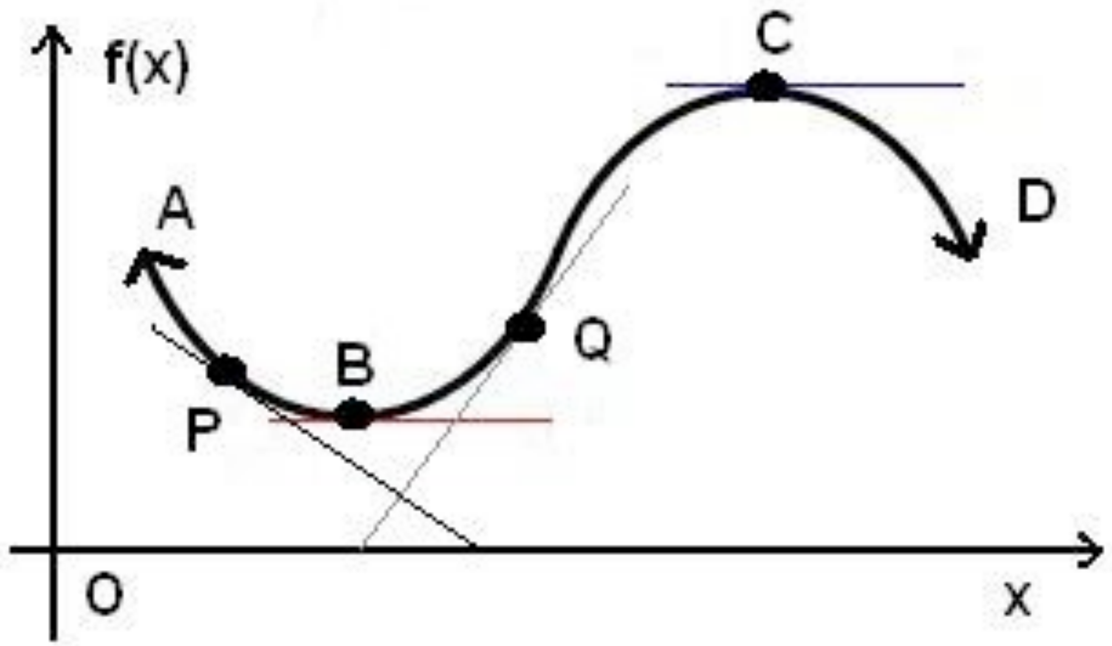
+ Smale complex / topological graph



Morse theory

Manifold	$\mathcal{M} \subset \mathbb{R}^n$
Function	$f : \mathcal{M} \to \mathbb{R}$
Critical point	$\mathbf{x} \in \mathcal{M}, \partial f(\mathbf{x}) = 0$
Index	$\#\{\lambda \underset{d}{\in} Eig(\partial^2 f), \lambda < 0\}$
Topology	$\chi = \sum_{i=0}(-1)^i \cdot m_i \ \ldots$

Function analysis is intuitive



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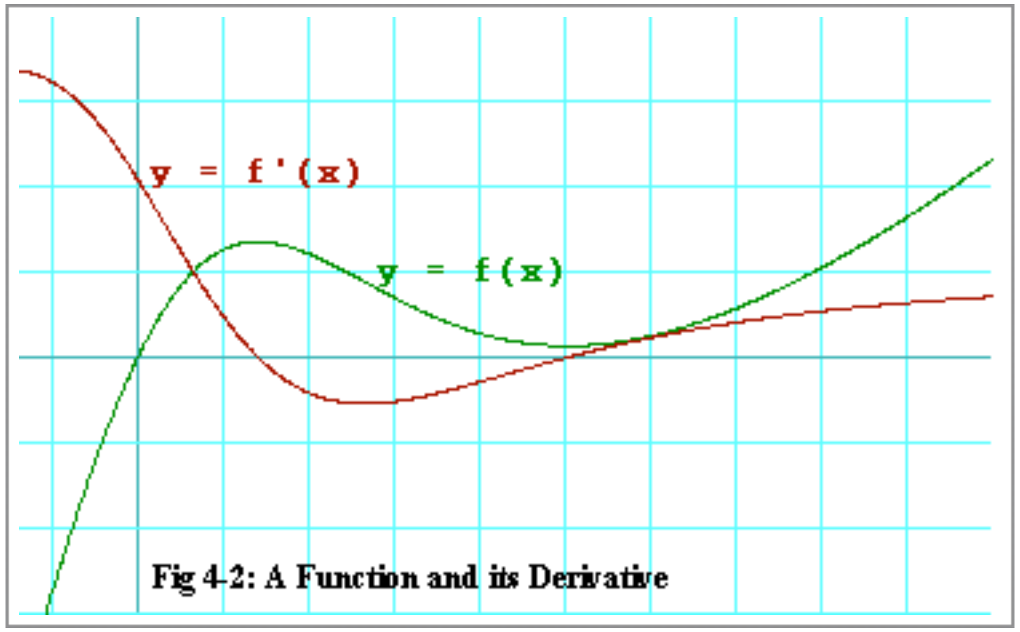

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Intuition:

quick and ready insight

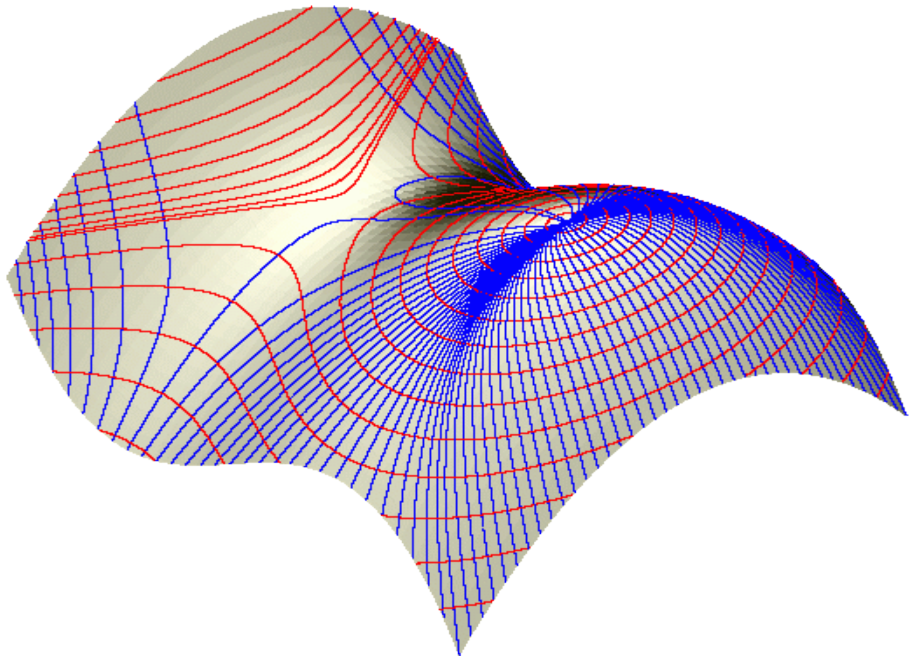
immediate apprehension or cognition

More topological objects

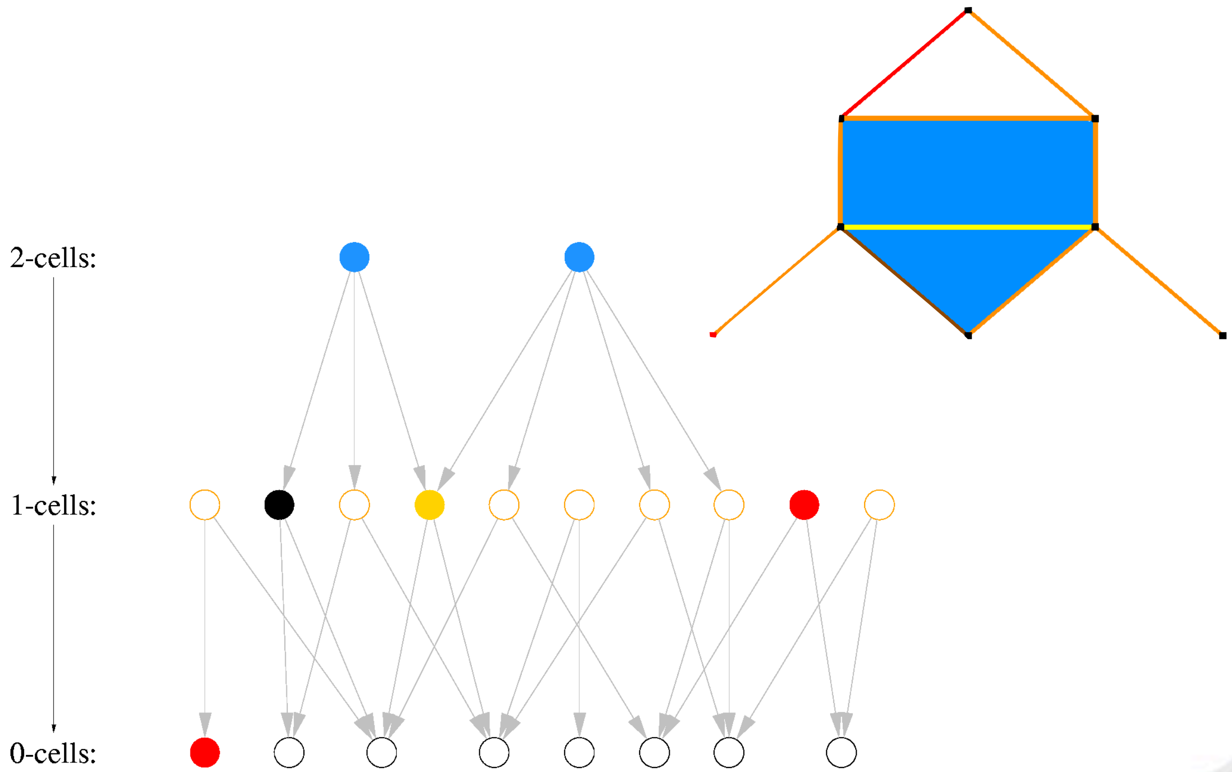
differential

discrete

Local function analysis in the discrete setting???

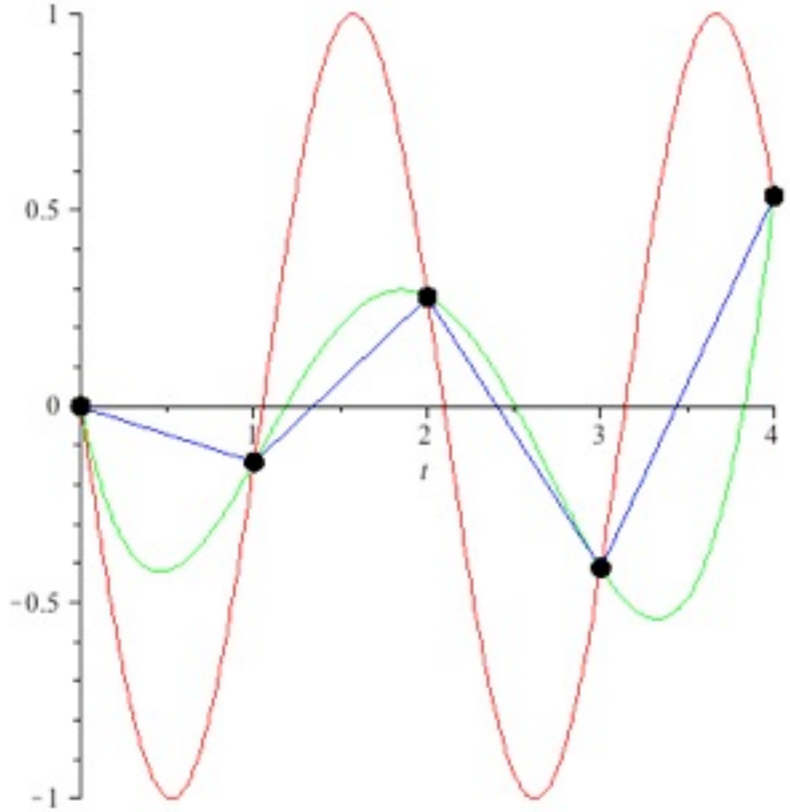


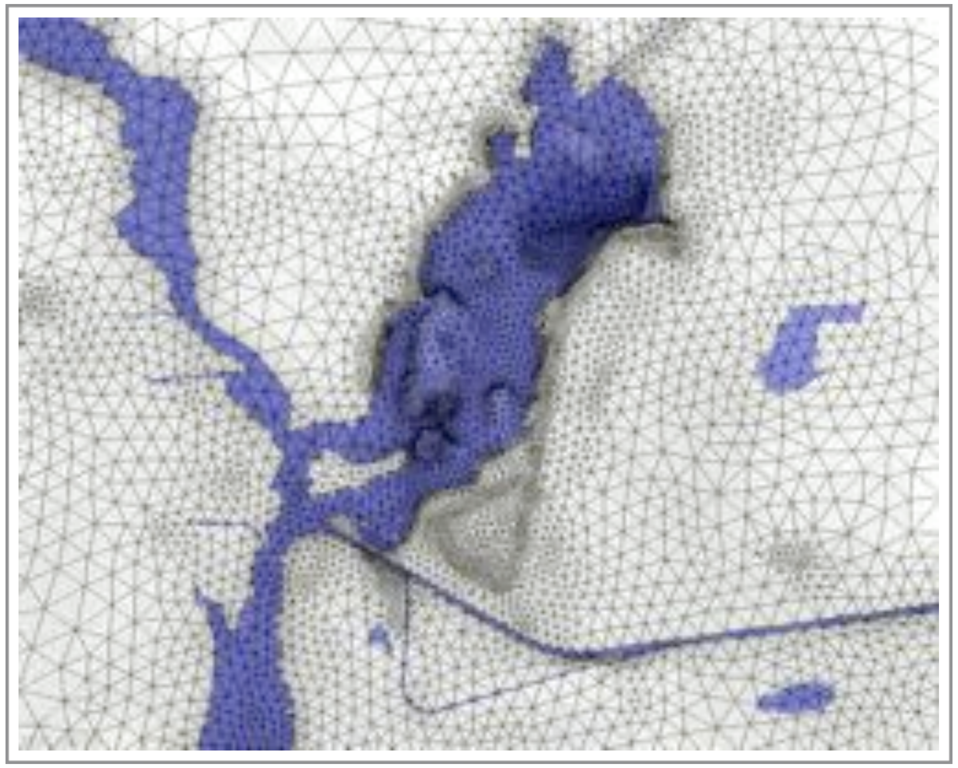
Sampled function
+ interpolation



Weight on
graph structures

Discrete function analysis?





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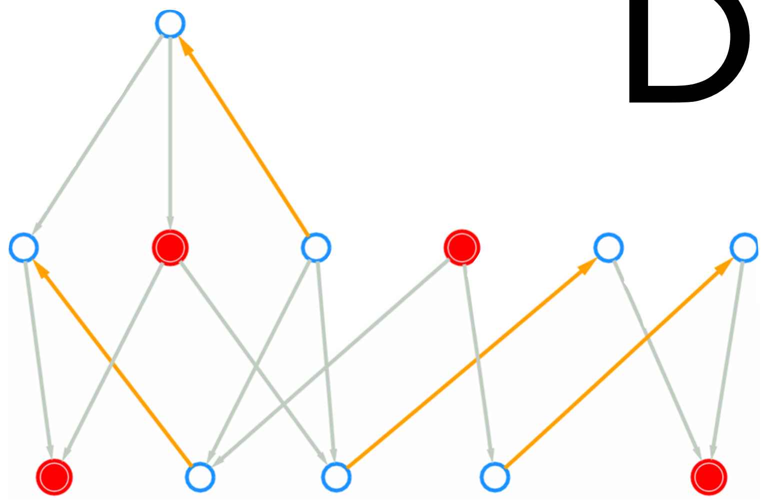
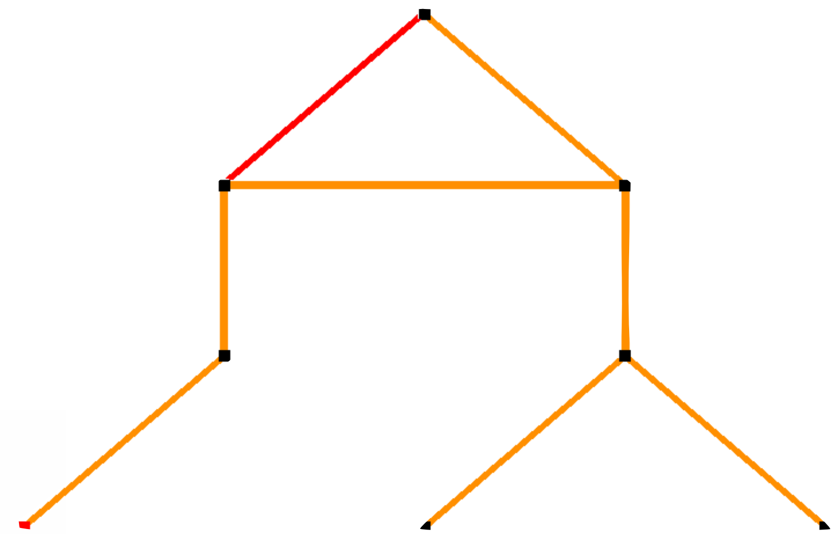
not intuitive:

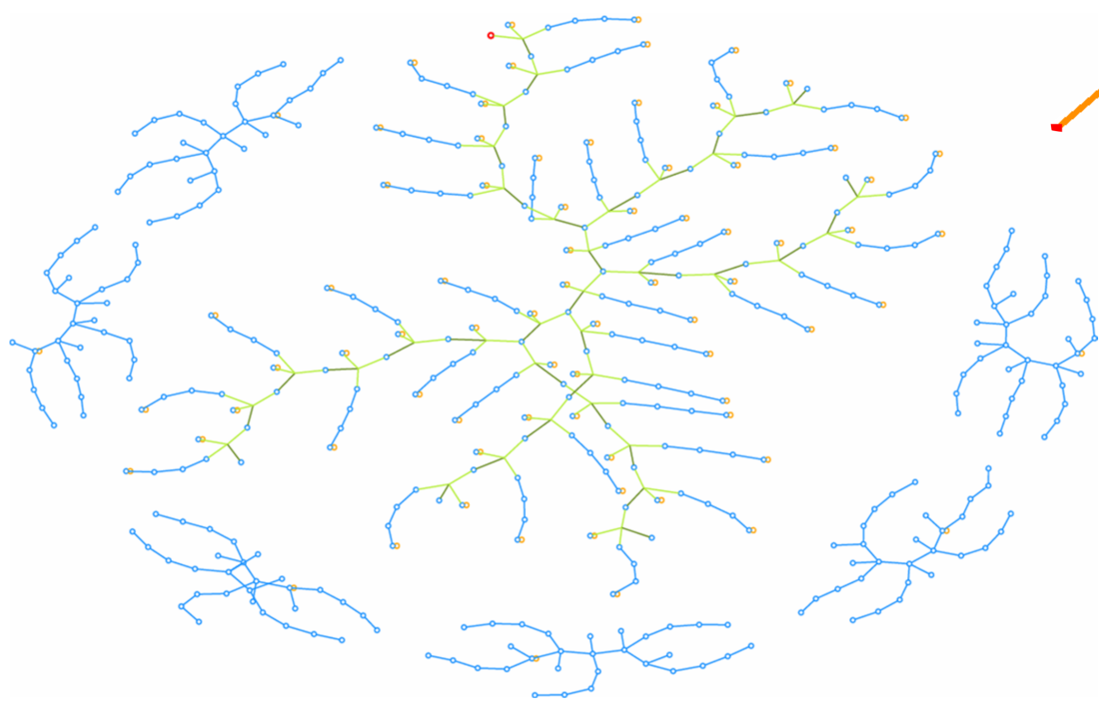
finite difference (polynomial interpolation)
Fourier derivative (harmonic interpolation)
finite elements (template approximation)

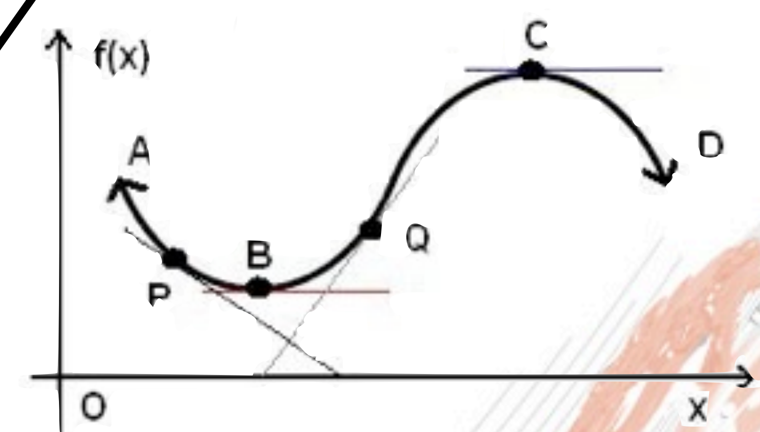
on manifold?

Discrete analysis?



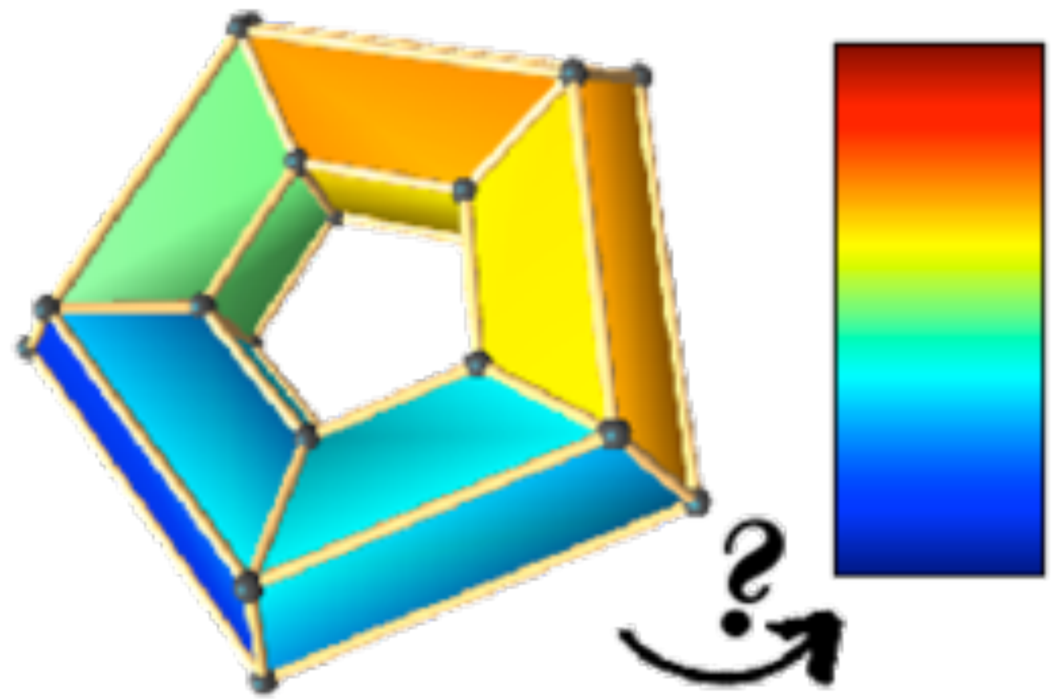






Intuition on graphs?

Discrete Morse theories



piecewise-linear interpolation

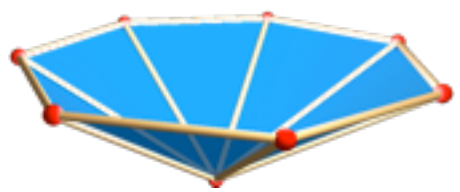
$\Rrightarrow$ Banchoff's approach

combinatorial formulation

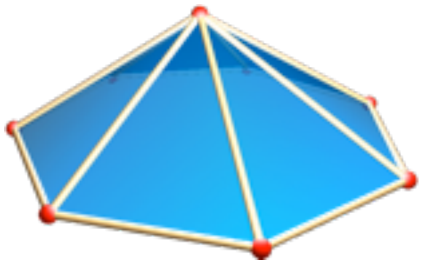
$\Rrightarrow$ Forman's approach

Banchoff’s approach


regular


minimum


saddle


maximum


degenerated

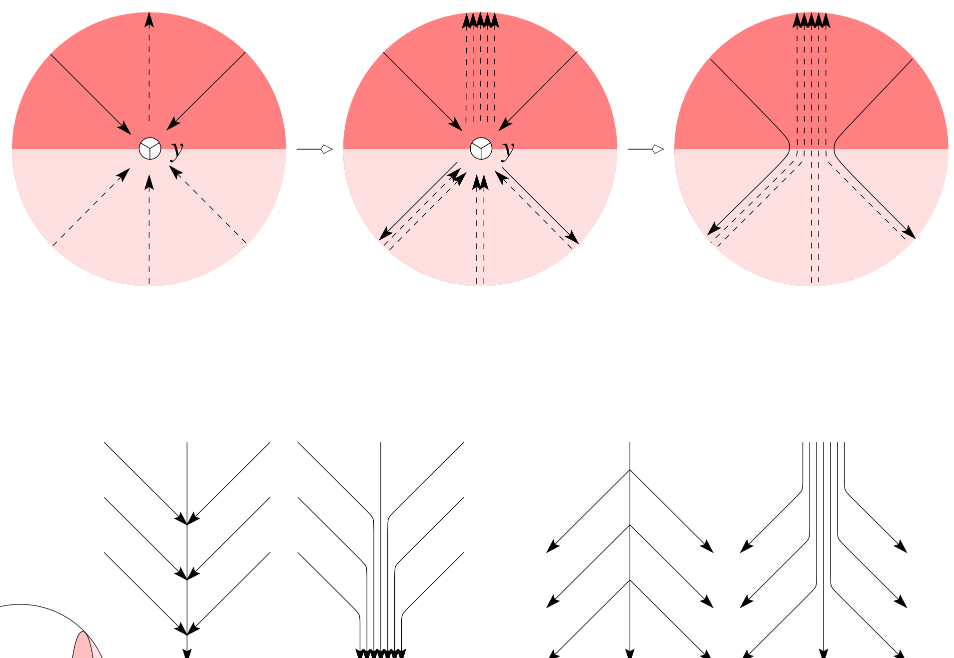
intuition: sampling of f

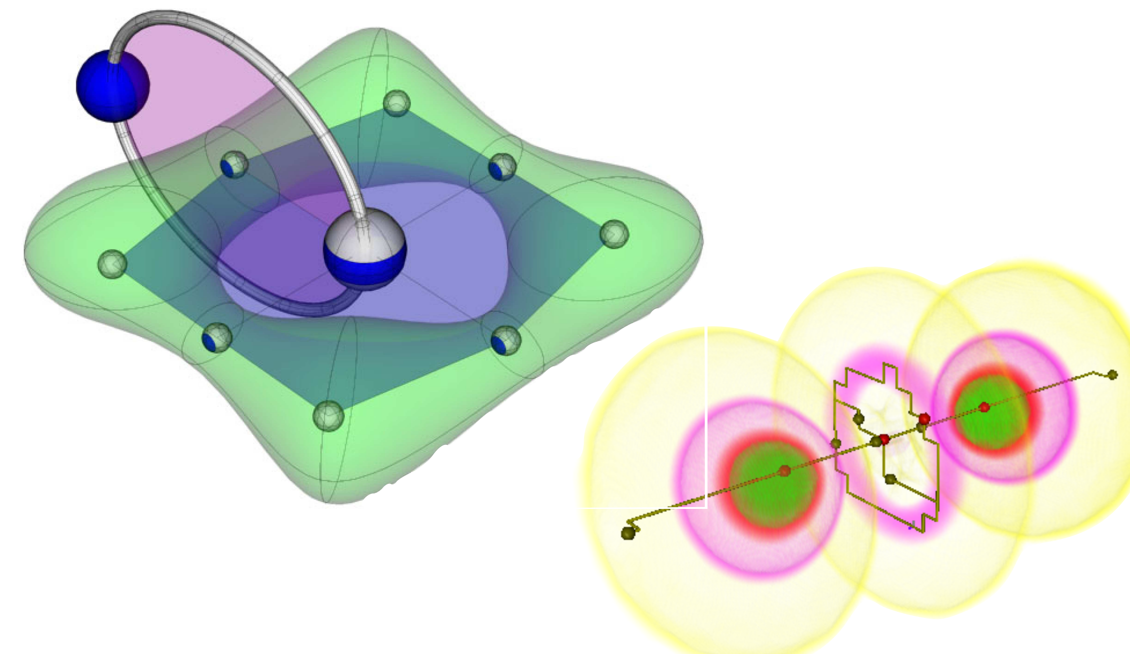
guarantees: Euler χ, Gauss Bonnet

local linear interpolations

$\Rrightarrow$ critical vertices

PL Morse-Smale complex





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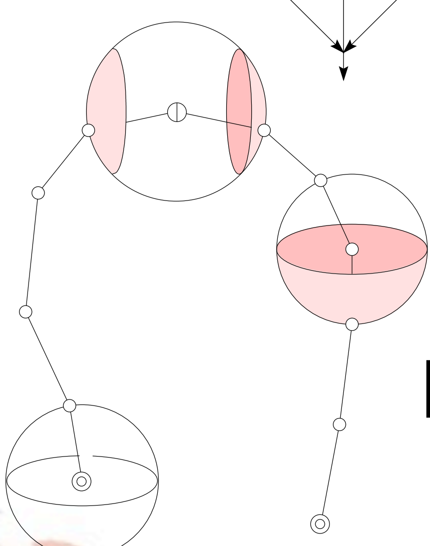
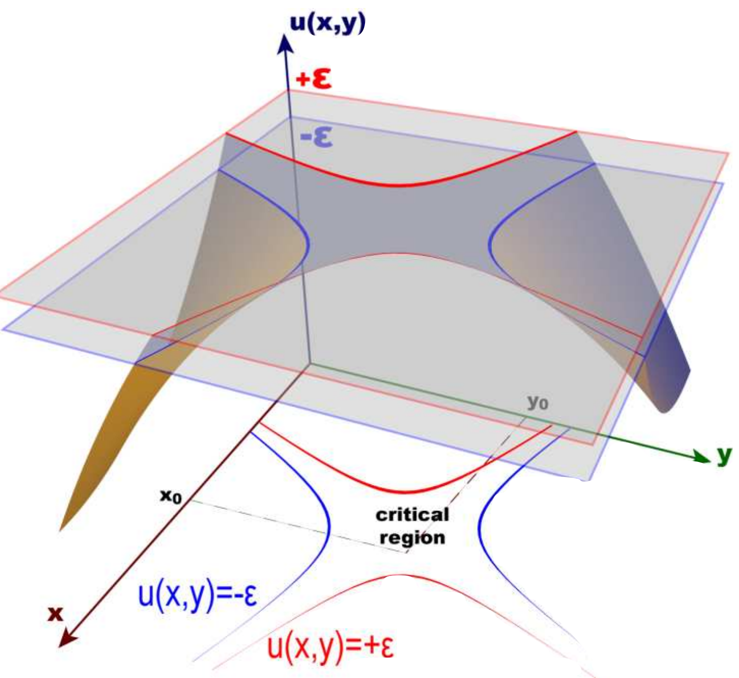


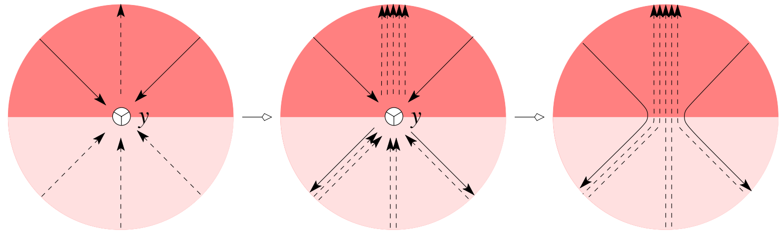
numerical integration of streamlines

$\Rrightarrow$ Morse-Smale structure

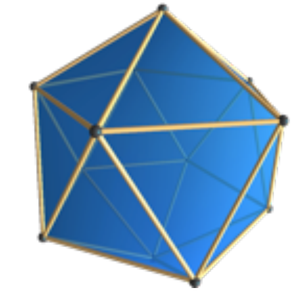


Morse theory

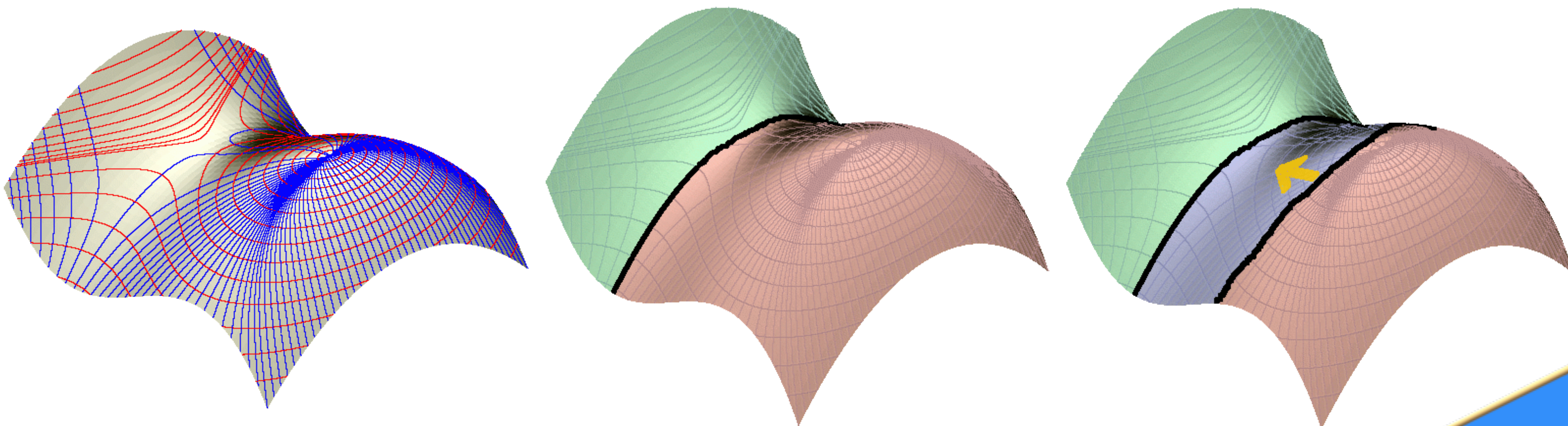
Object	$\mathcal{M} \subset \mathbb{R}^n$
Function	$f : \mathcal{M} \to \mathbb{R}$
Critical set	$\partial f^{-1}(\{0\})$
Index	$\#Eig(\partial^2 f) \cap \mathbb{R}_-$
Topology	$\chi = \sum (-1)^i \cdot m_i \ \ldots$



∂, PL - Morse theories

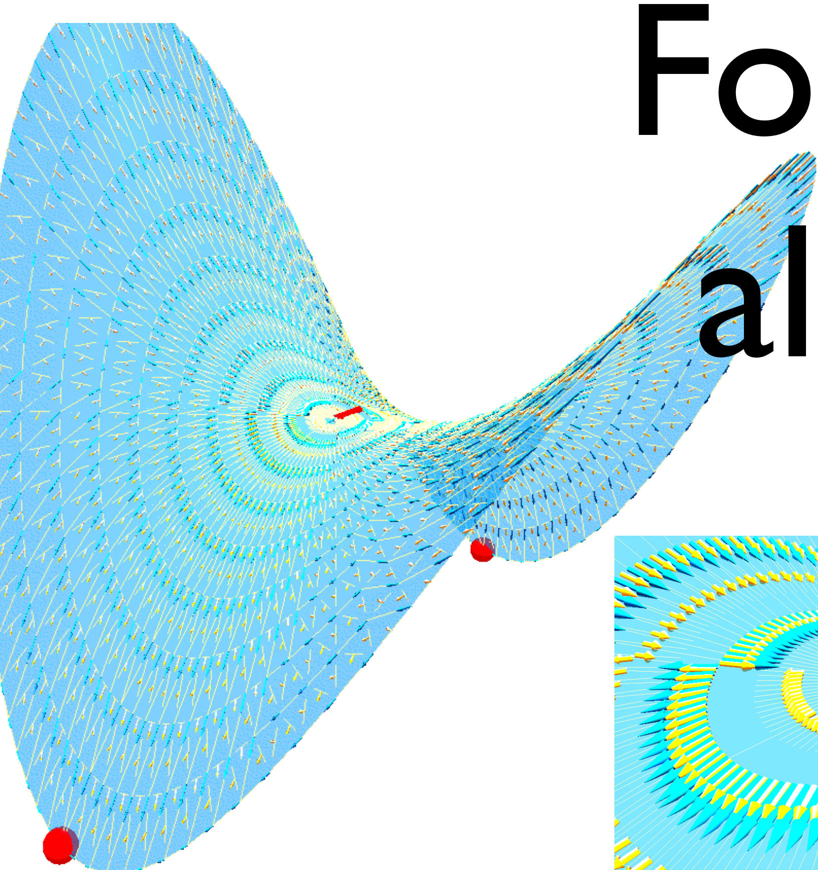
	∂	PL
Object	$\mathcal{M} \subset \mathbb{R}^n$	$\lvert\mathcal{K}\rvert \subset \mathbb{R}^n$ 
Function	$f : \mathcal{M} \to \mathbb{R}$	$f : \mathcal{K}_0 \hookrightarrow \mathbb{R}$
Critical set	$\partial f^{-1}(\{0\})$	
Index	$\#Eig(\partial^2 f) \cap \mathbb{R}_-$	*0* *1..n-1* *n*
Topology	$\chi = \sum (-1)^i \cdot m_i \ \ldots$	χ

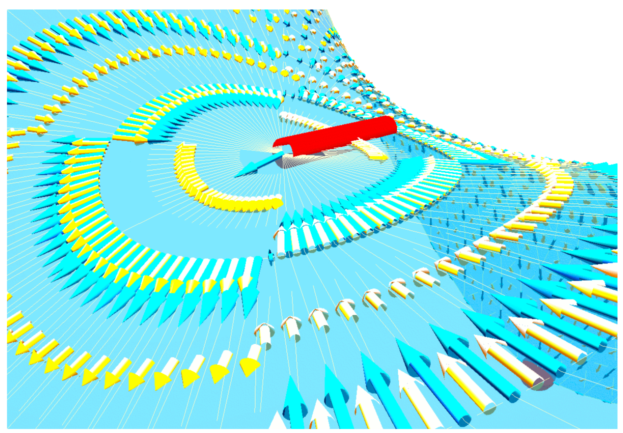
Forman's approach: differential topology view

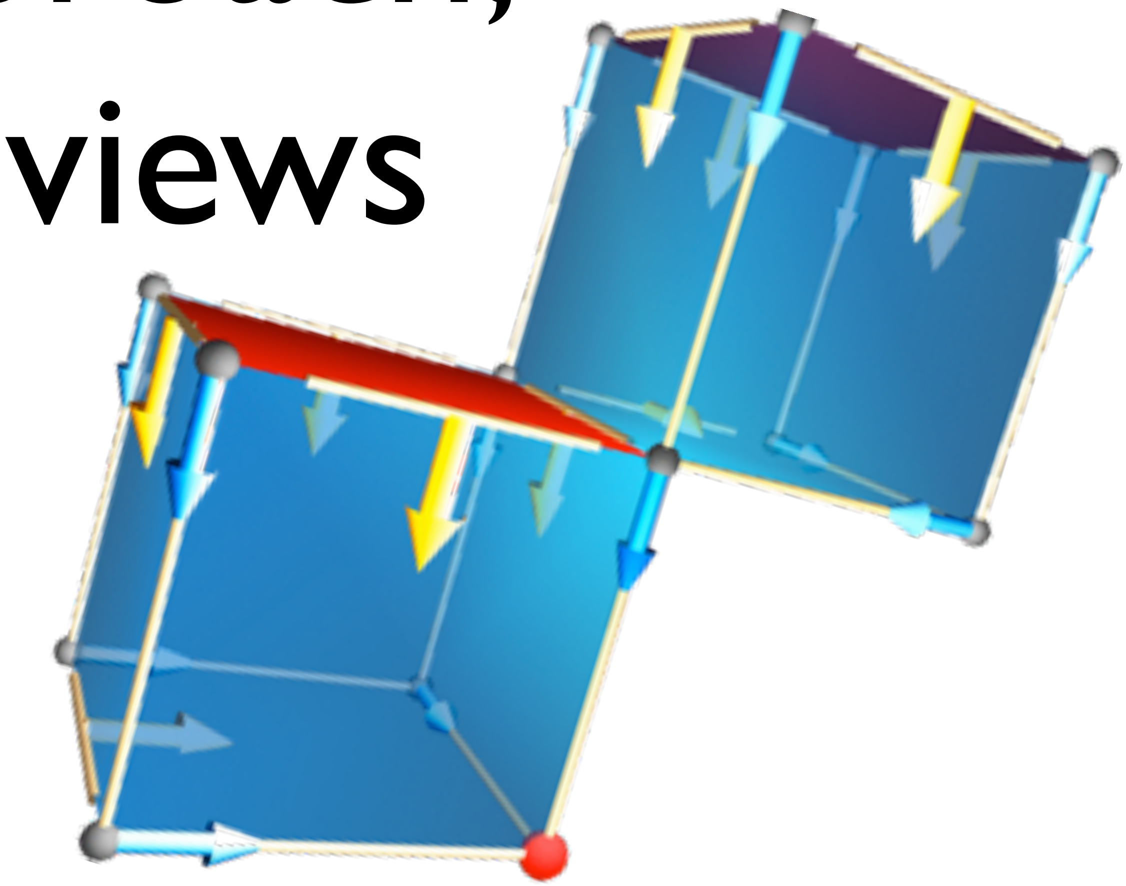


Start from Morse-Smale complex
Subdivide to reach your complex
arrows ⇛ gradient vector field

Forman’s approach, algorithmic views







matching

combinatorial field

$\Rrightarrow$ matching along the flow

$\Rrightarrow$ critical = unmatched

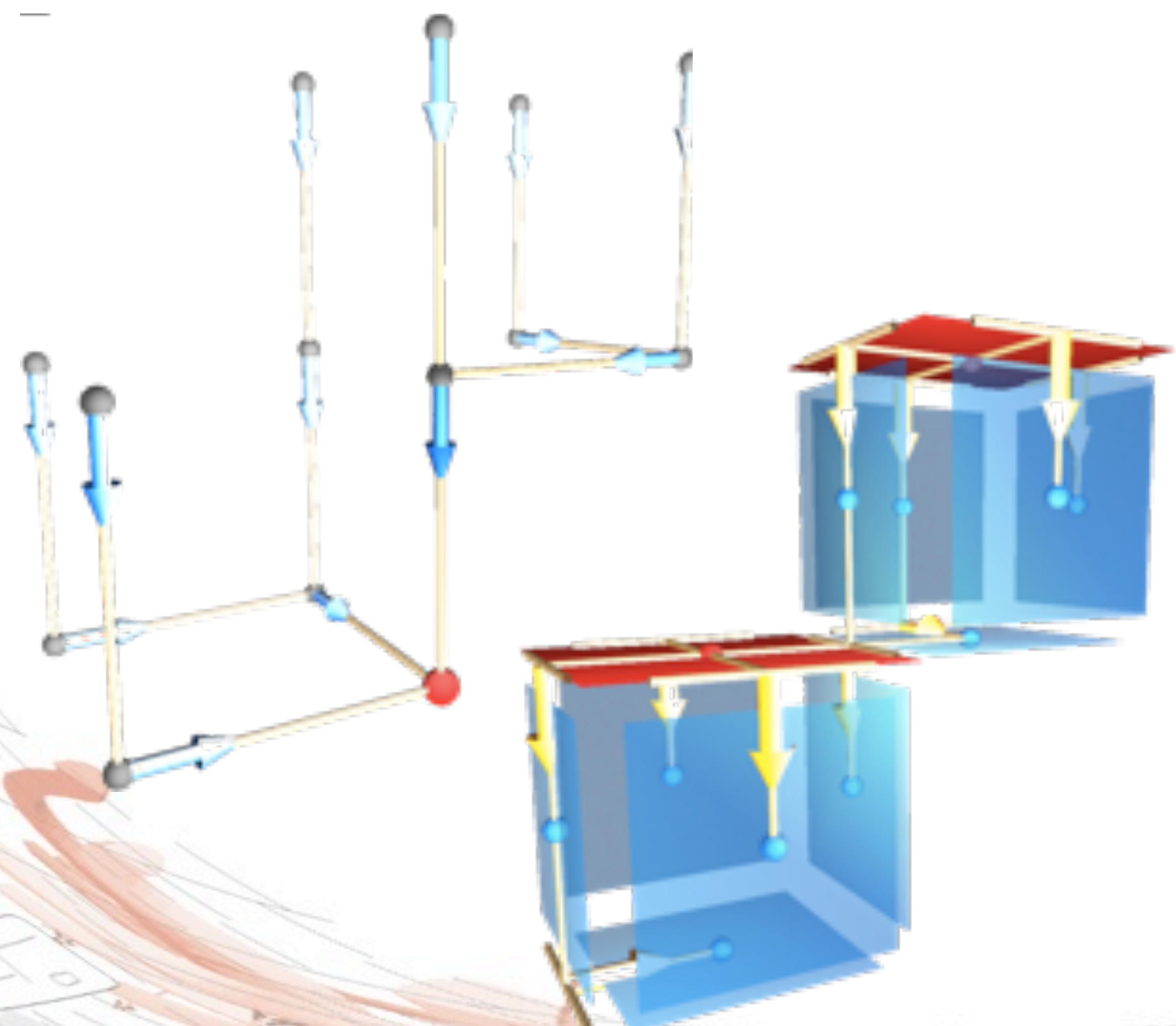
tree

gradient field

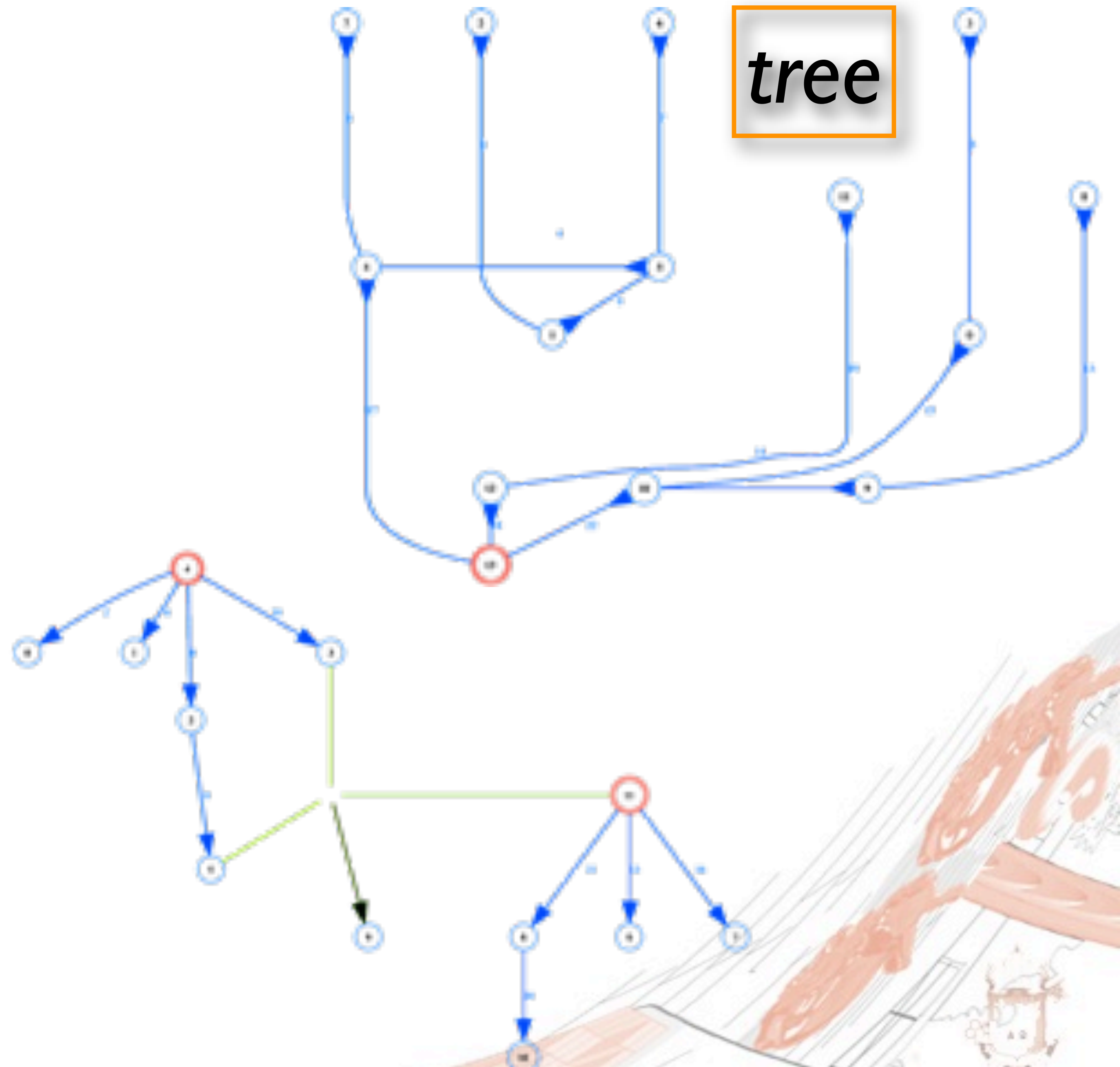
no closed gradient path $\Rrightarrow$ acyclic

Acyclic matching

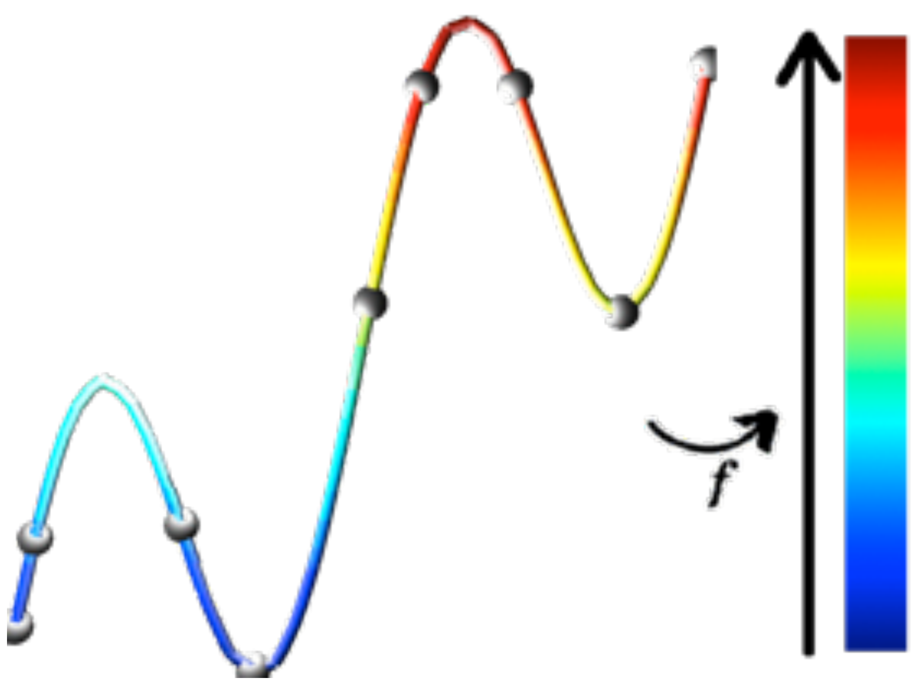
matching

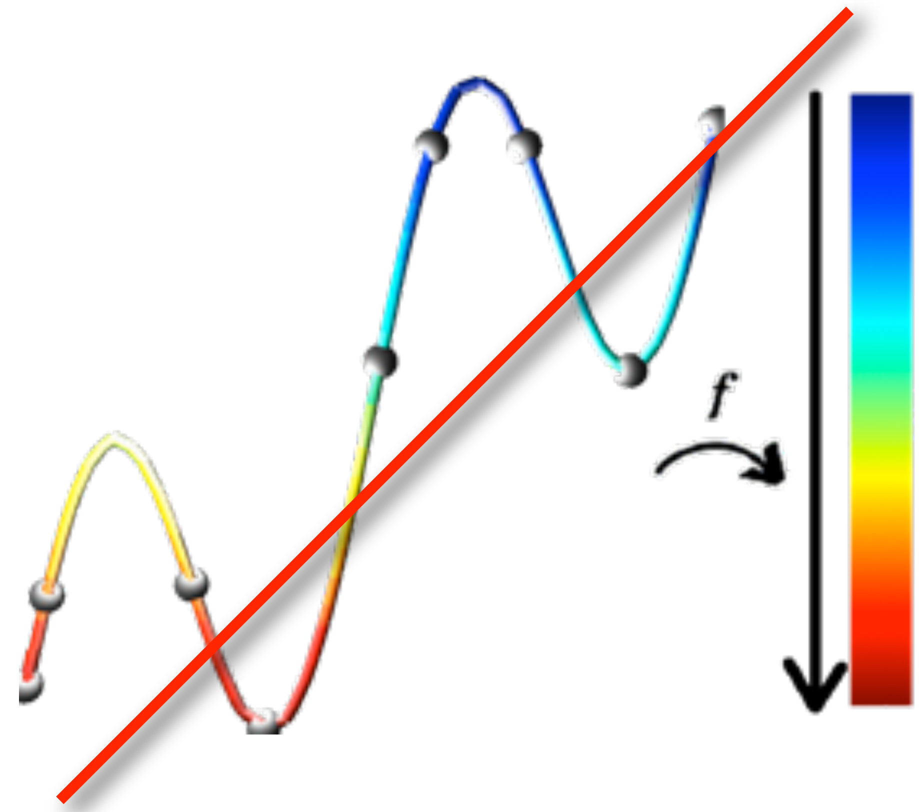


tree



Forman's approach: differential topology view

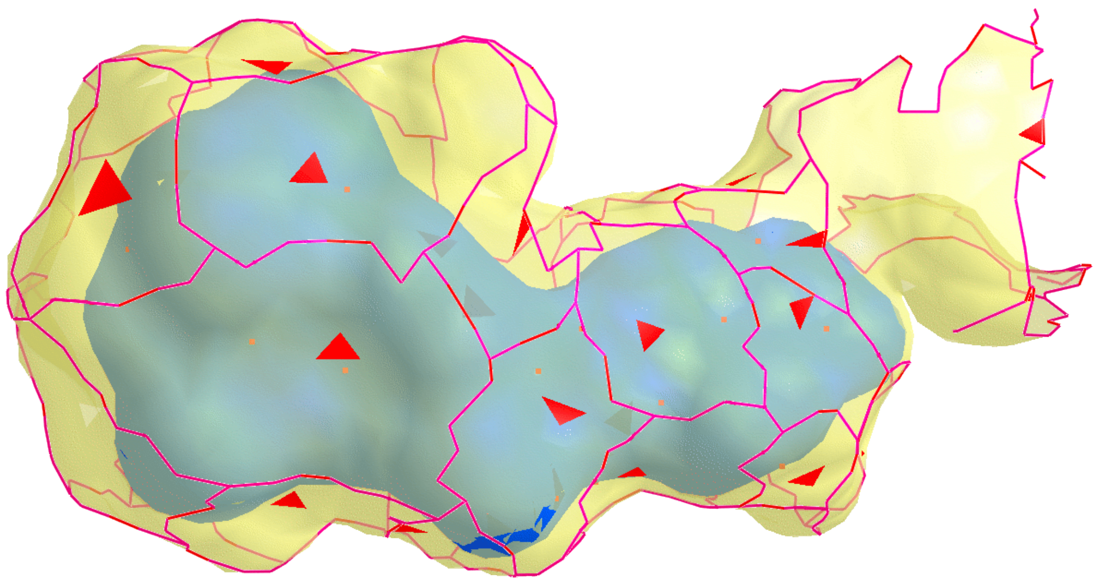




guarantees: Morse inequalities, nD, homotopy, Witten homology...

intuition without differential function?

Discrete Smale Compelx



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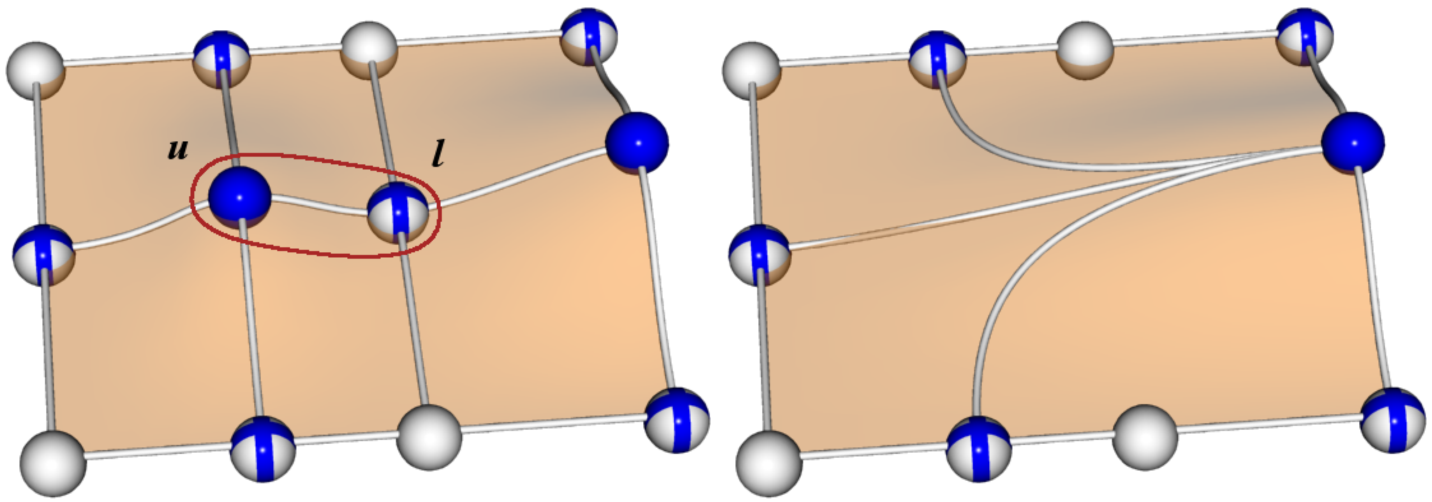

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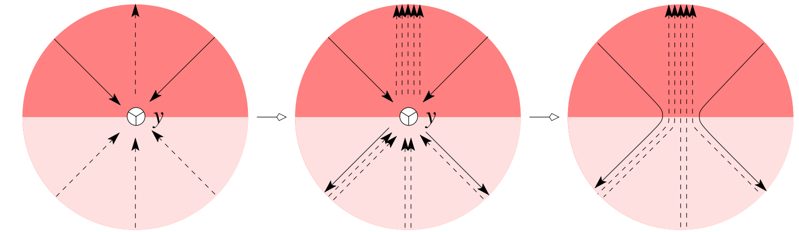
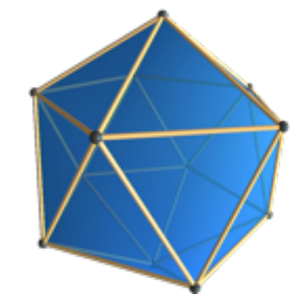
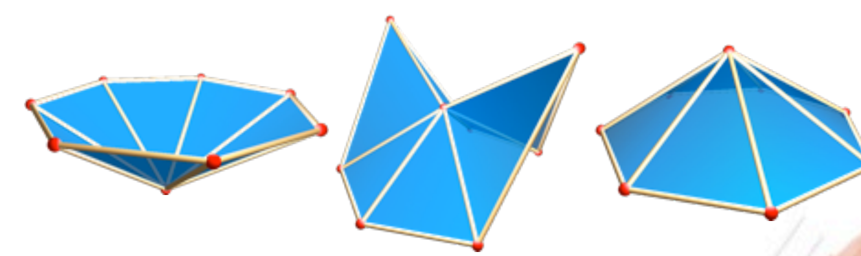
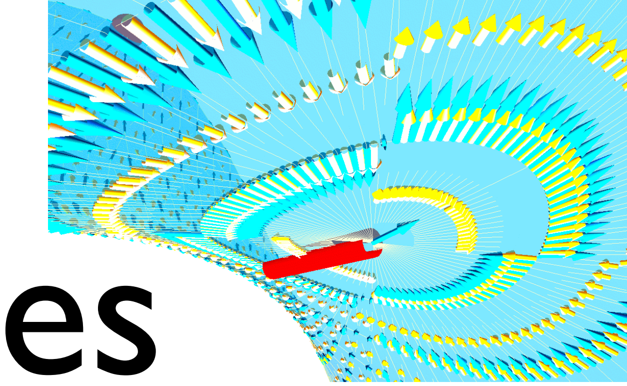
from critical cells

from cancellations

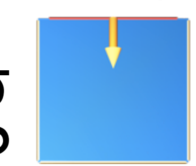


∂, PL - Morse theories

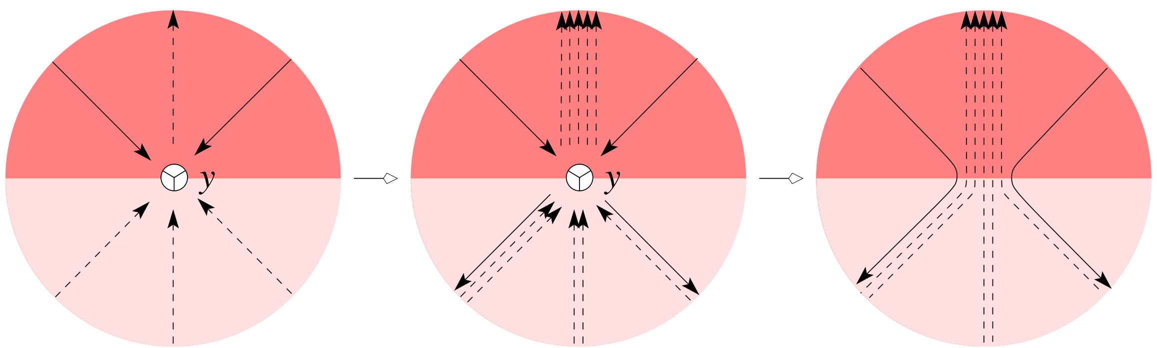
	∂	PL
Object	$\mathcal{M} \subset \mathbb{R}^n$	$\lvert\mathcal{K}\rvert \subset \mathbb{R}^n$ 
Function	$f : \mathcal{M} \to \mathbb{R}$	$f : \mathcal{K}_0 \hookrightarrow \mathbb{R}$
Critical set	$\partial f^{-1}(\{0\})$	
Index	$\#Eig(\partial^2 f) \cap \mathbb{R}_-$	*0* *1..n-1* *n*
Topology	$\chi = \sum(-1)^i \cdot m_i \ \ldots$	χ

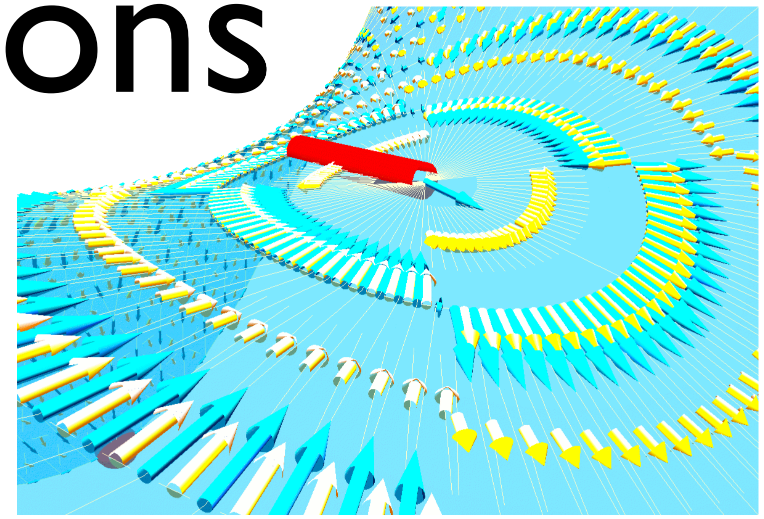


∂, Forman - Morse theories

	∂	**Forman**  
Object	$\mathcal{M} \subset \mathbb{R}^n$	CW complex $\mathcal{K}$ 
Function	$f : \mathcal{M} \to \mathbb{R}$	acyclic matching 
Critical set	$\partial f^{-1}(\{0\})$	unmatched cells
Index	$\#Eig(\partial^2 f) \cap \mathbb{R}_-$	cell dimension
Topology	$\chi = \sum (-1)^i \cdot m_i \ \ldots$	$\chi = \sum (-1)^i \cdot m_i \ \ldots$

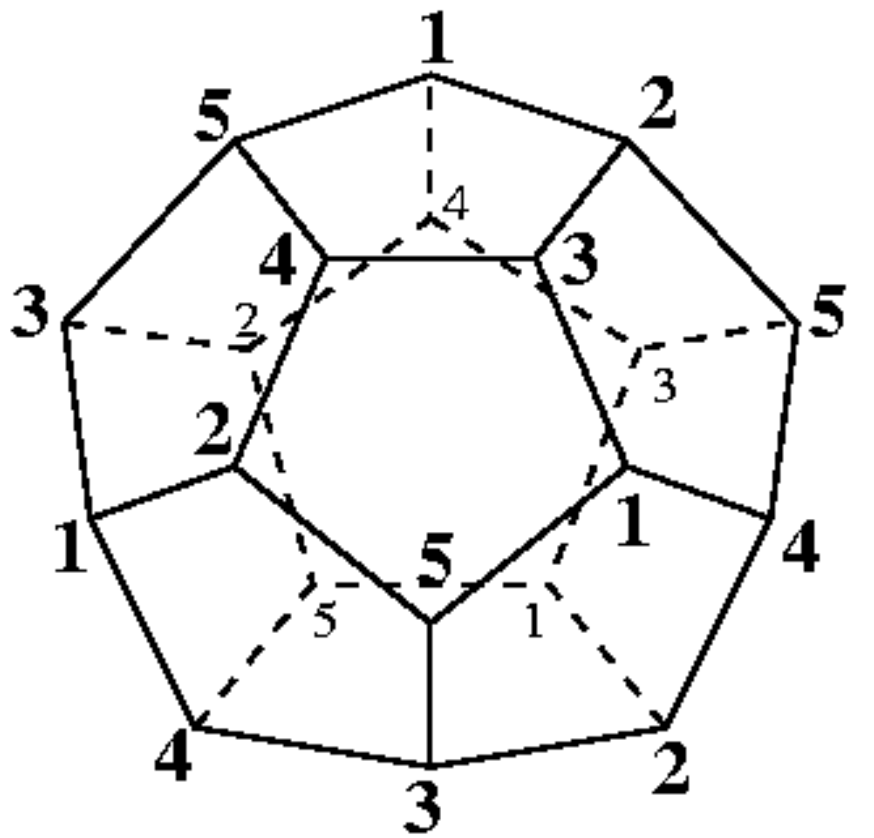
A *priori* pros and cons

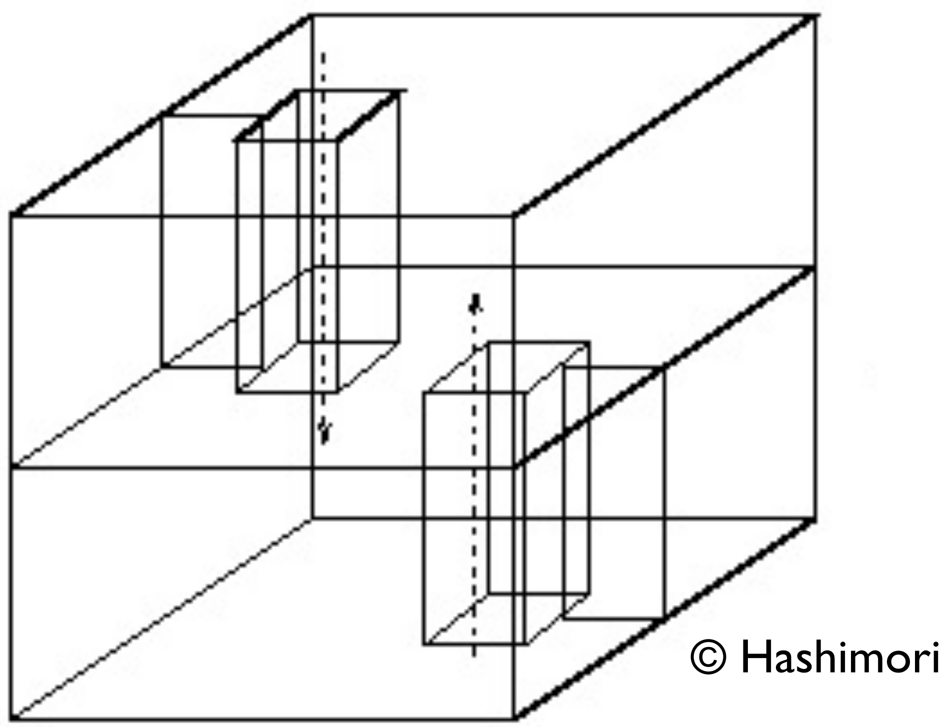




Banchoff's approach	*Forman's approach*
+ intuitive / geometric	+ correct global topology
+ controlled critical set	+ robust (graph algorithms)
+ many numerical tools	+ efficient (combinatorial)
- nD guarantees?	- critical set localization?
- global robustness?	- graph intuition?

Some monster cases

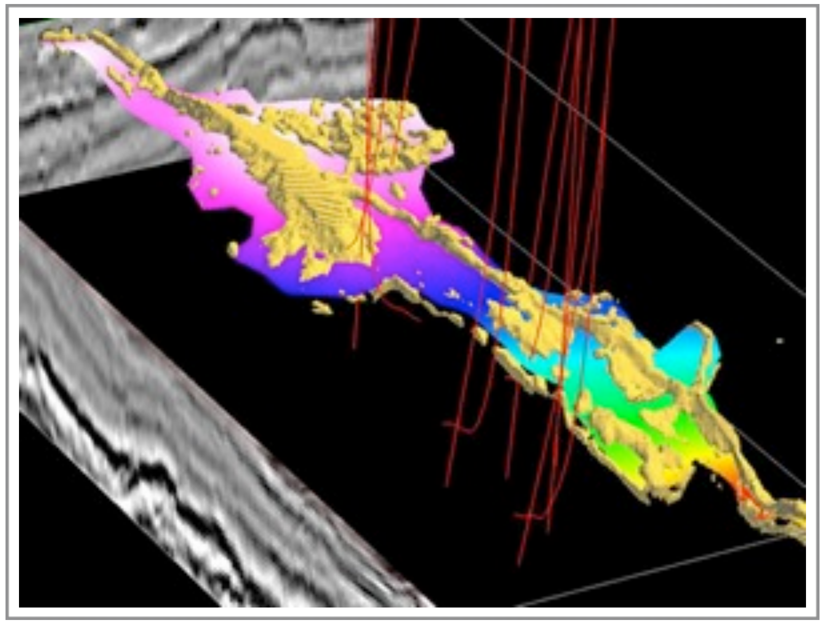


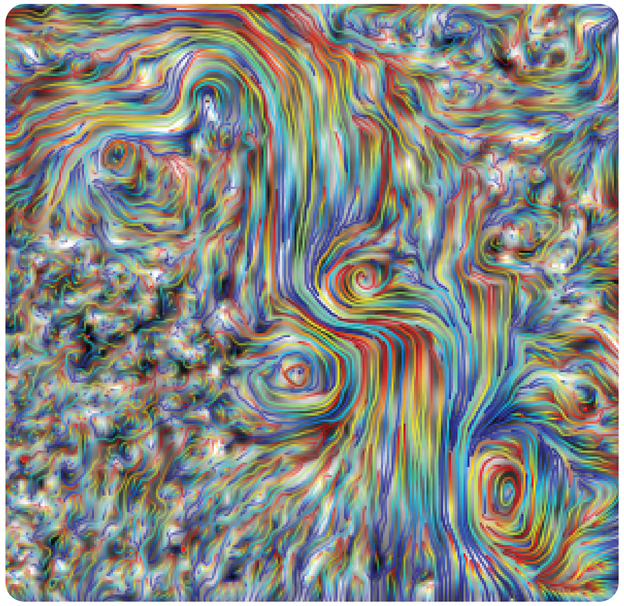


Poincaré's
homological sphere
(missing critical points)

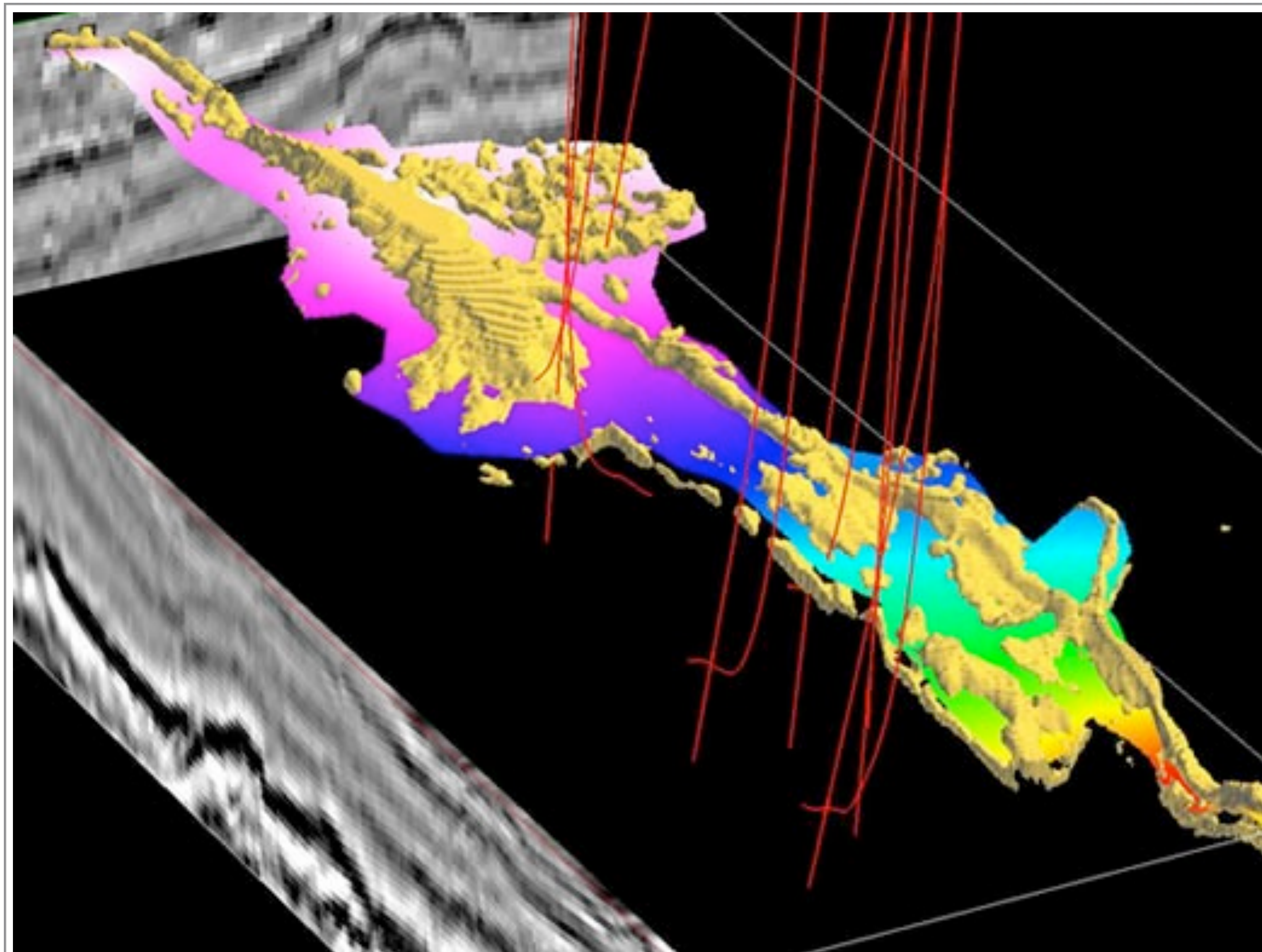
Bing's
house of 2 rooms
(extra critical cell)

Learning from examples



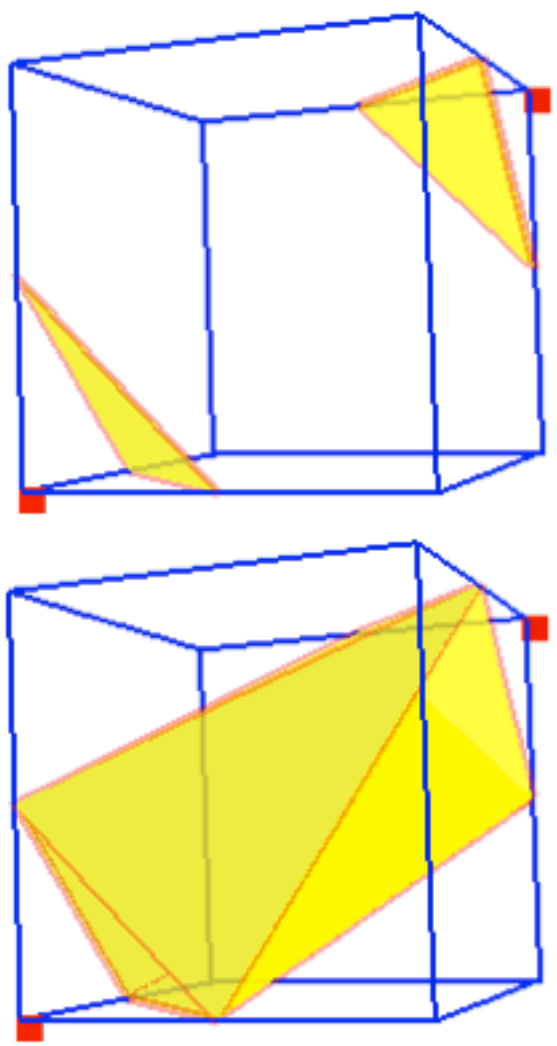


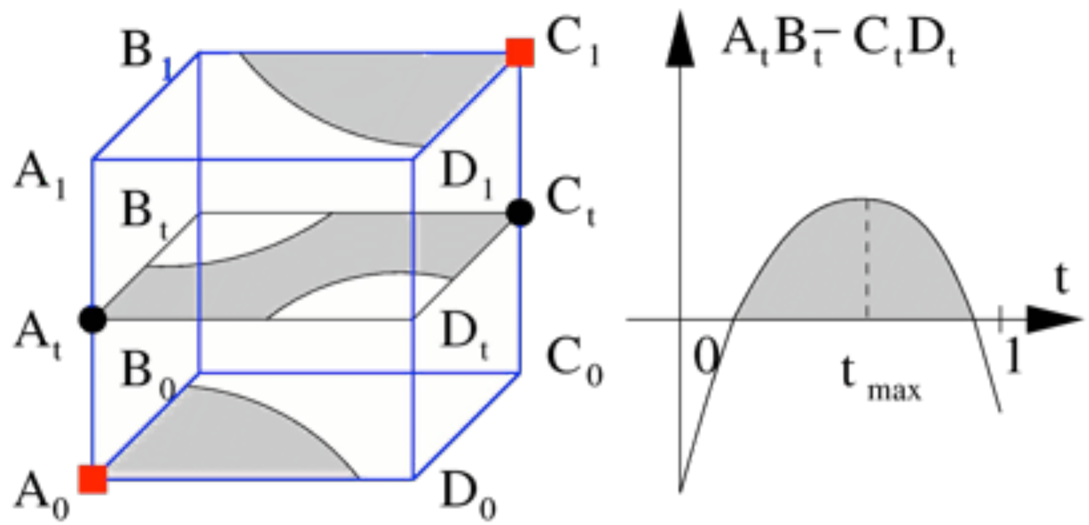
Isosurface extraction

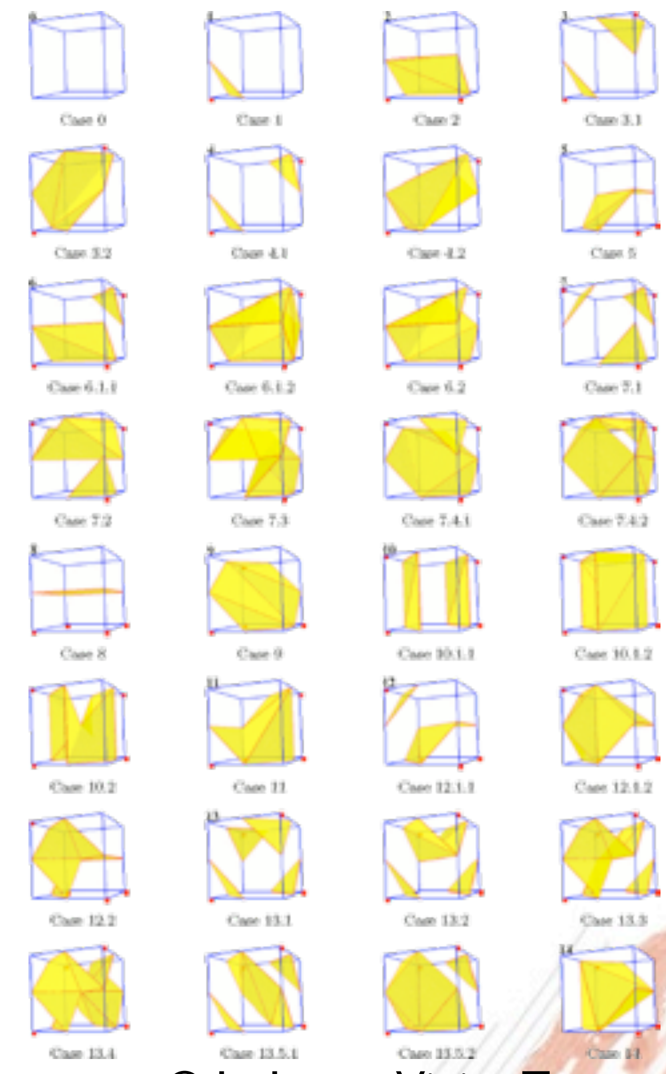


© Petrobras

Isosurface extraction



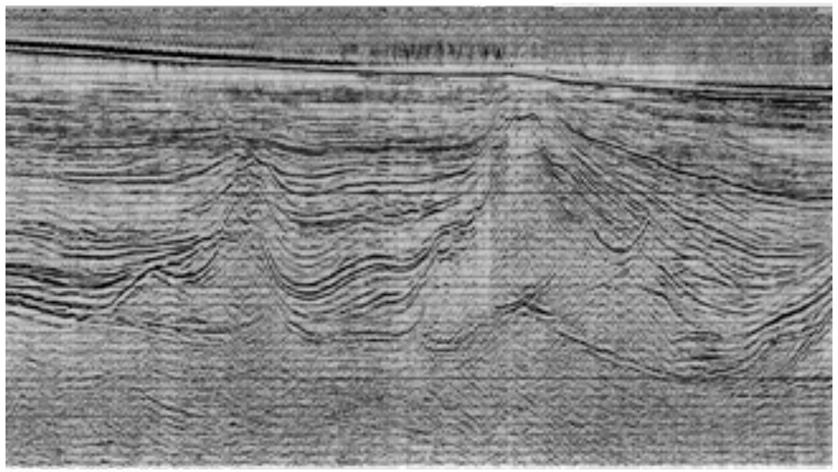


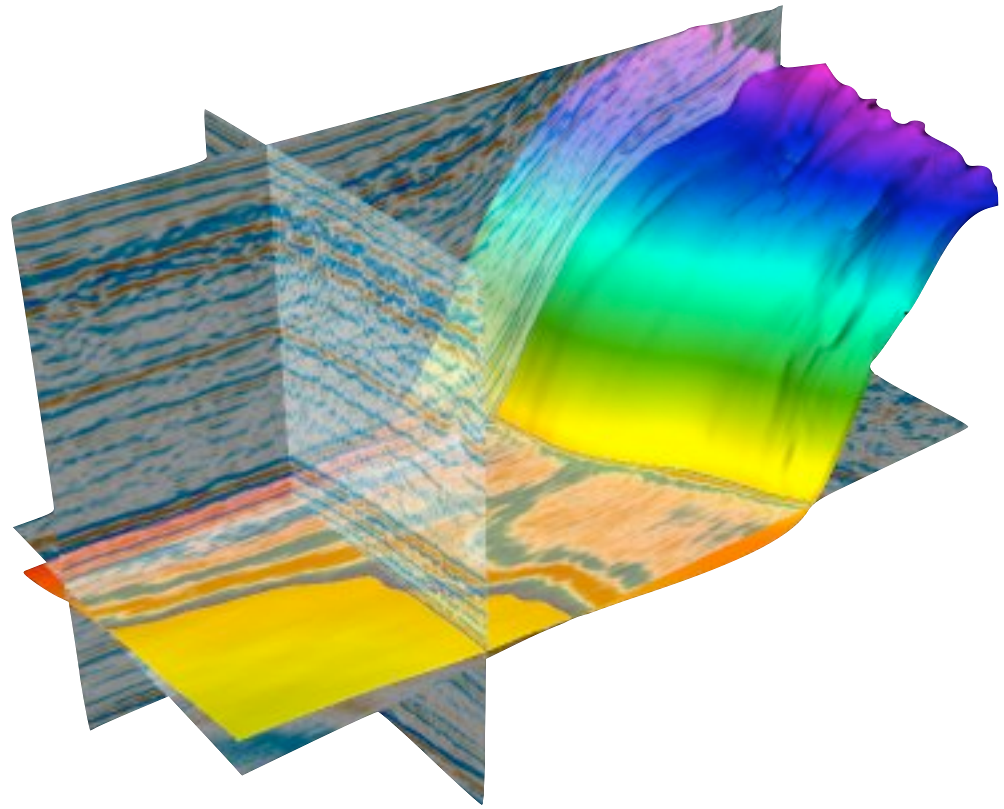


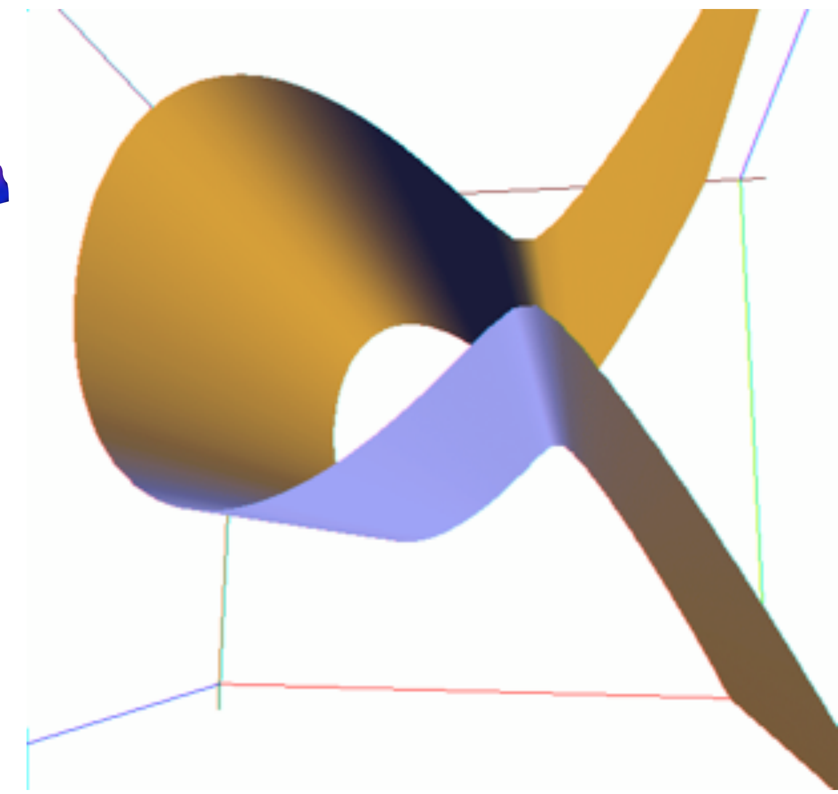
Topological cases of Marching Cubes

$\Rrightarrow$ differentiable function analysis

Large isosurface topology





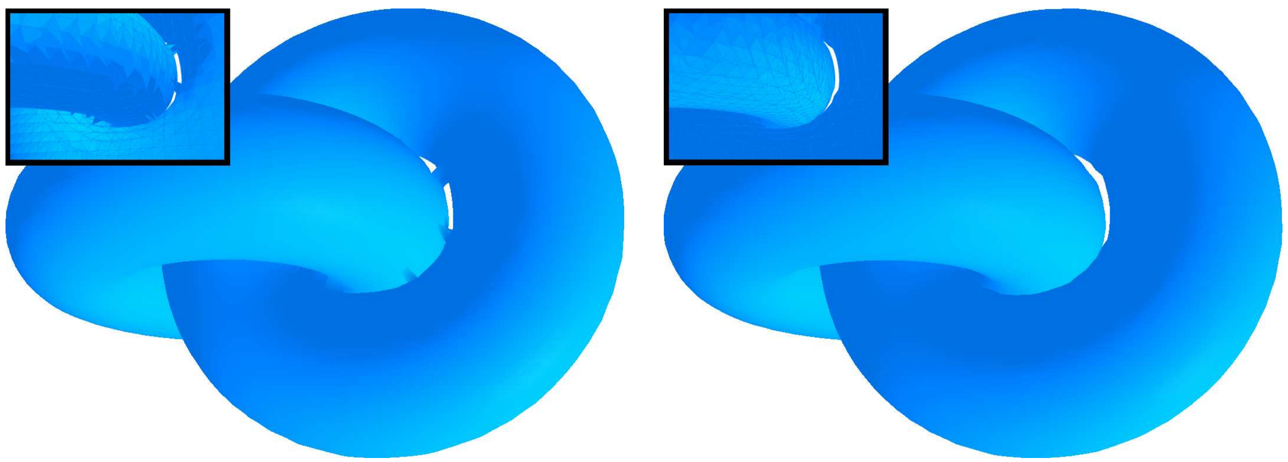


Topology without the isosurface

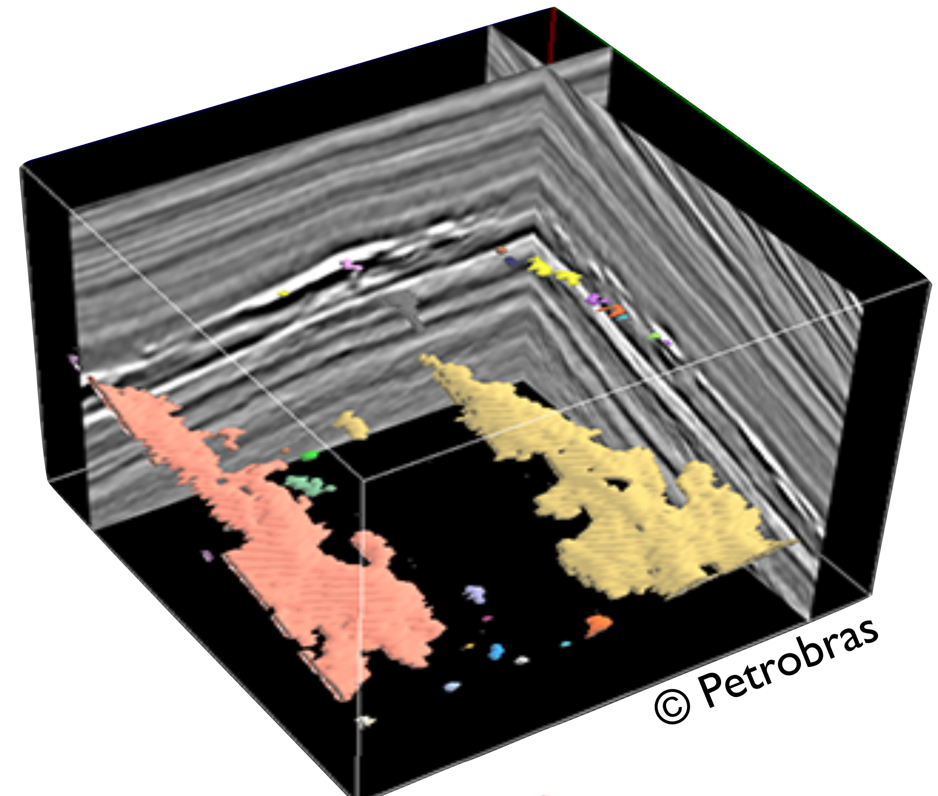
Mid-scale control and filtering

global + efficiency ⇛ Forman's line

Some Isosurfaces' Topology



© L., Lopes, Vieira, Tavares



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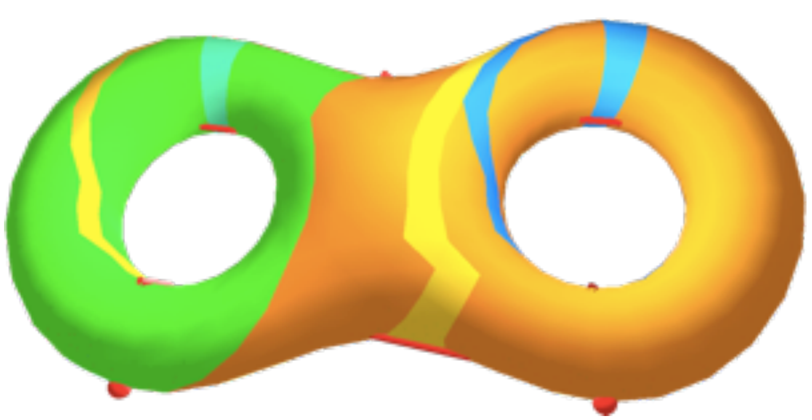
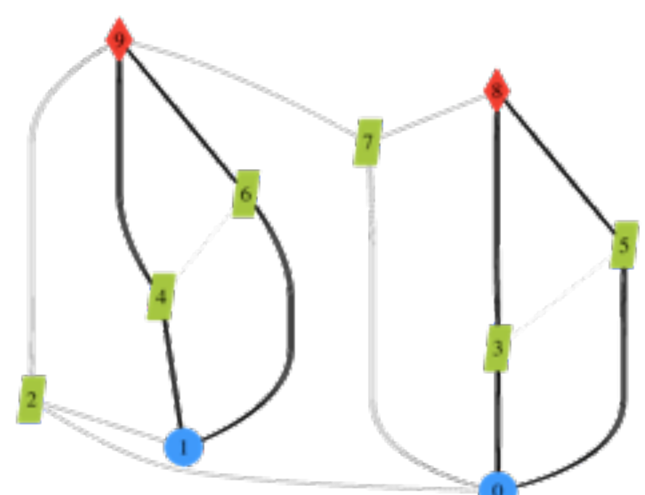
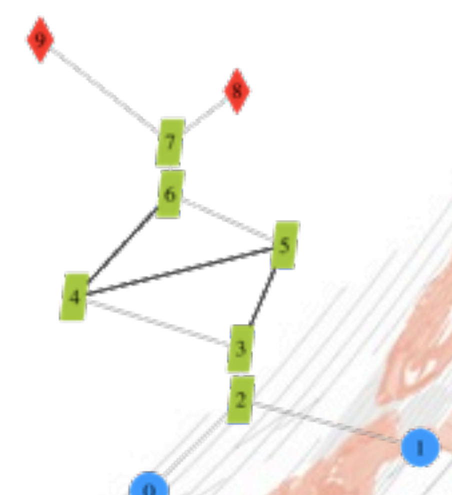


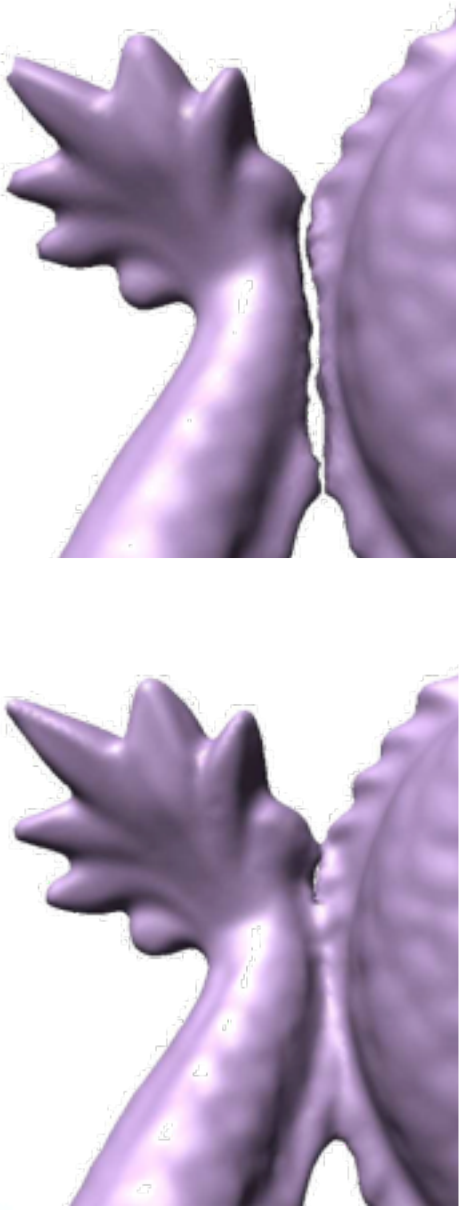


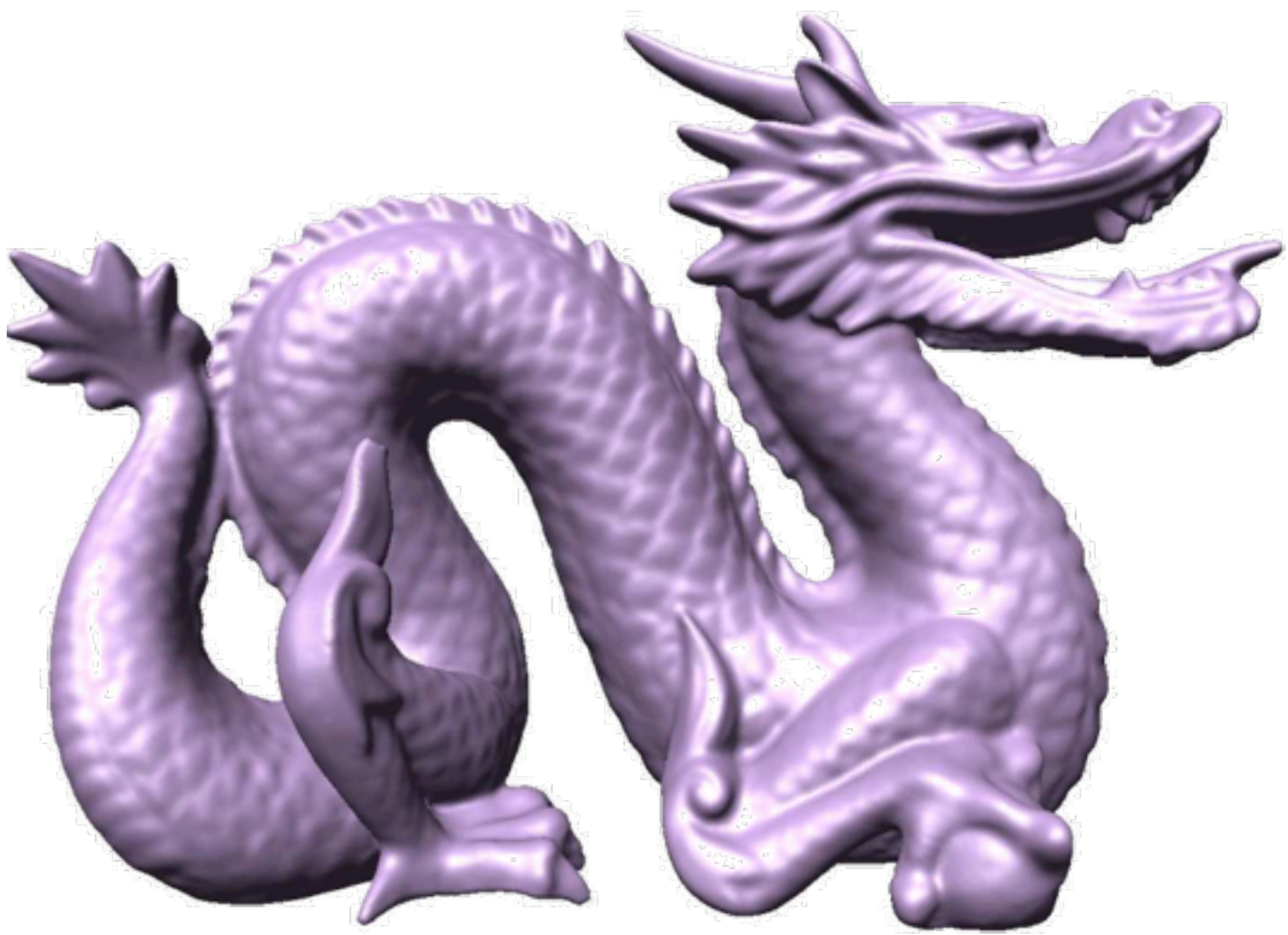
Smale complex



Reeb graph

Surface reconstruction





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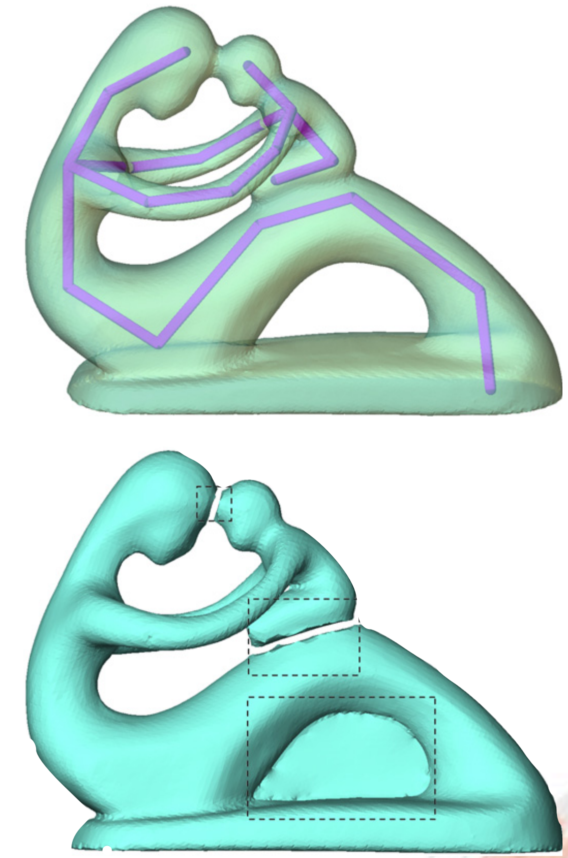
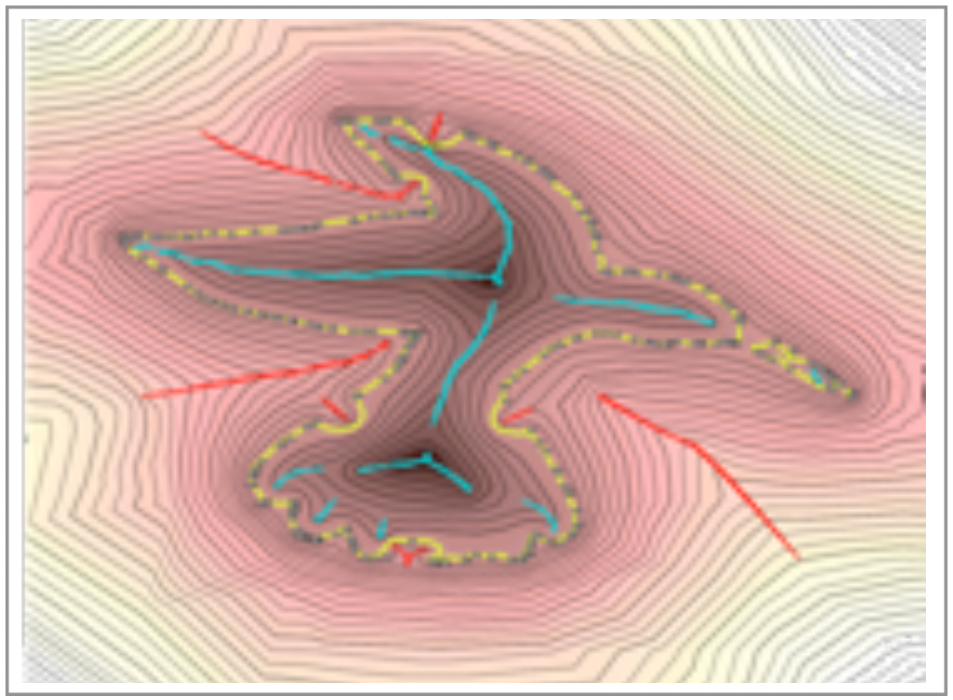


© Ju, Zhou, Hu

noisy, sparse point set

$\Rrightarrow$ correct topology?

Surface reconstruction



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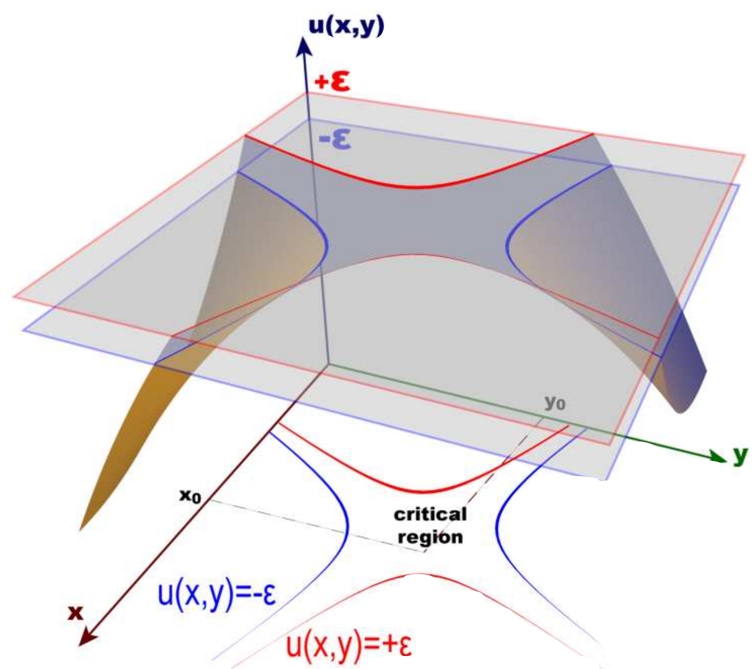
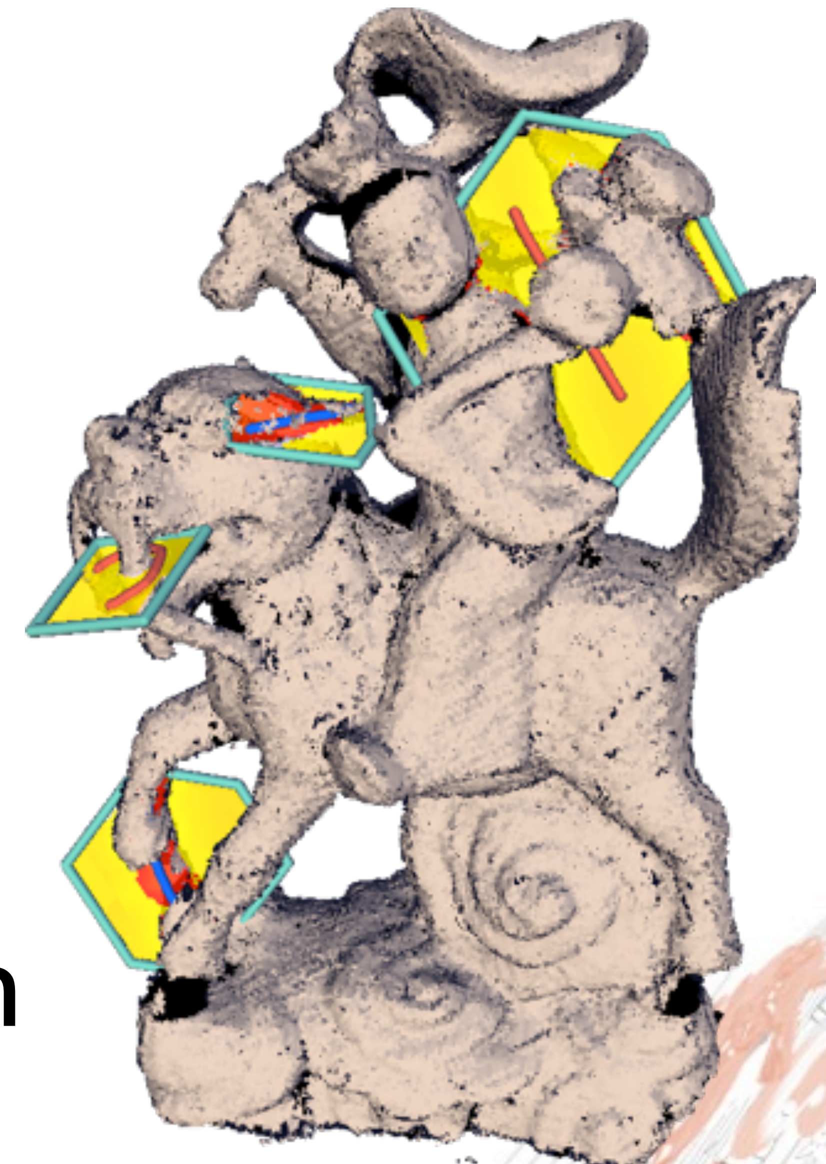


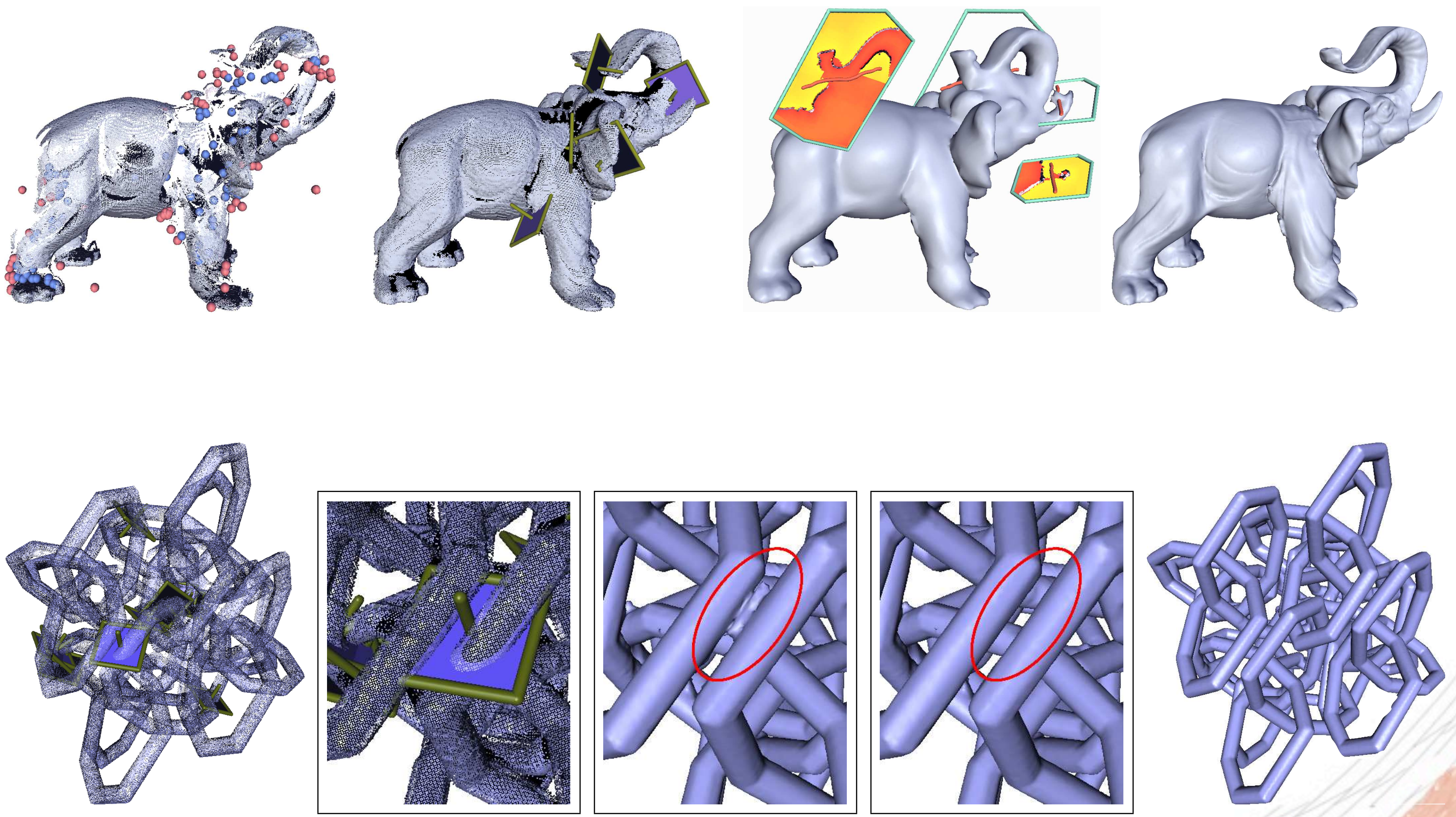


interactive topology edition

local critical regions

⇛ Banchoff's line

Topology-aware reconstruction



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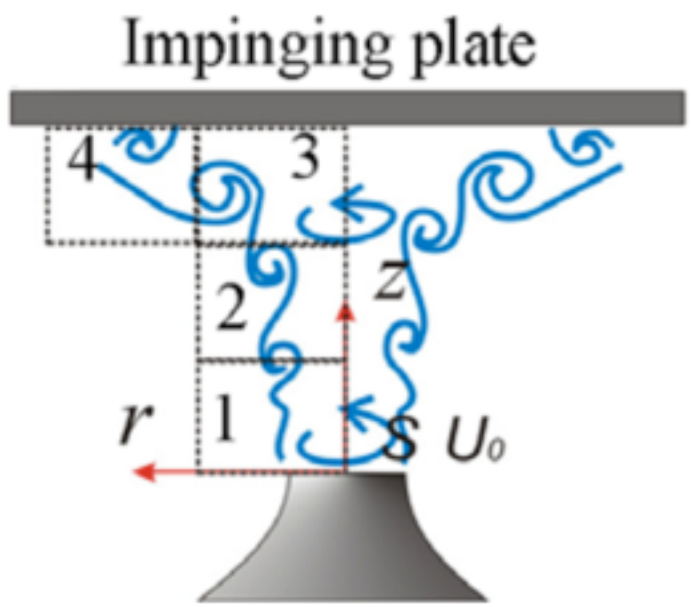
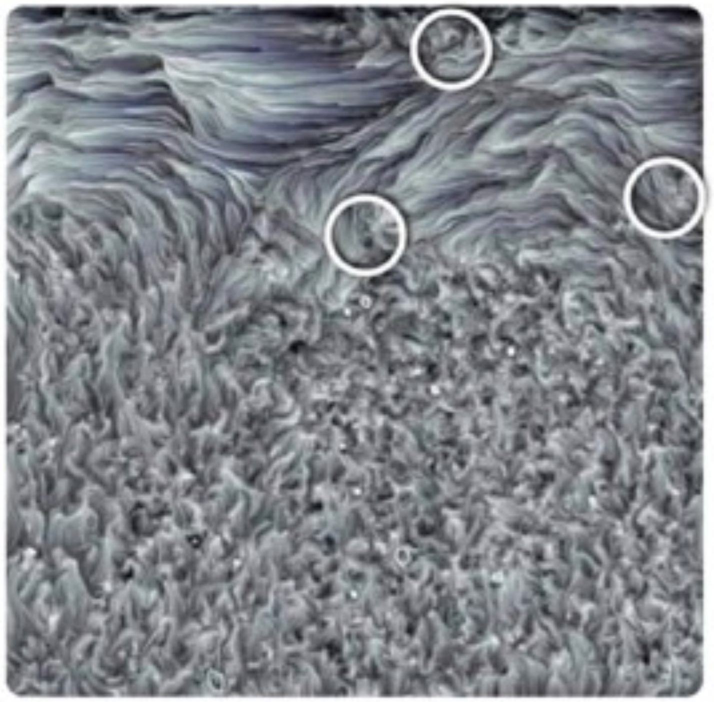
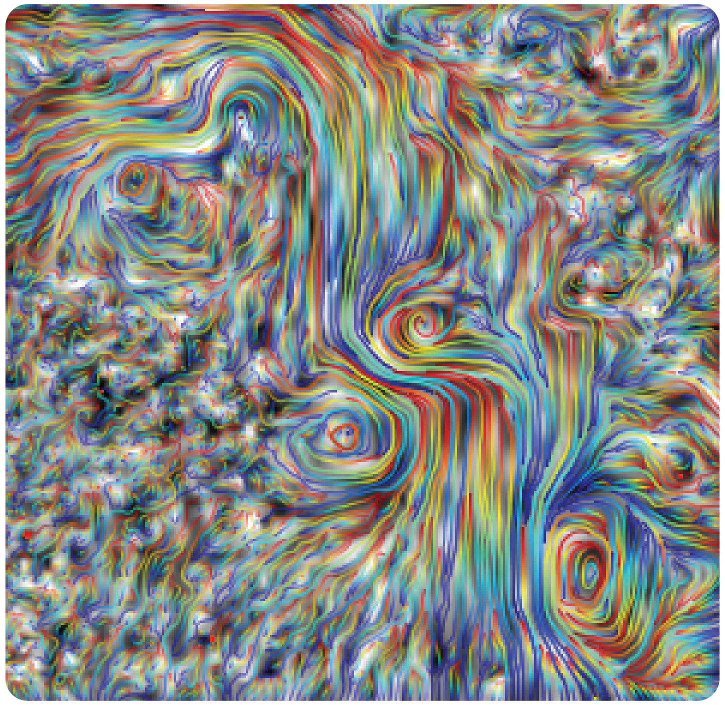
Vector field de-noising



Mechanical Dept, PUC-Rio





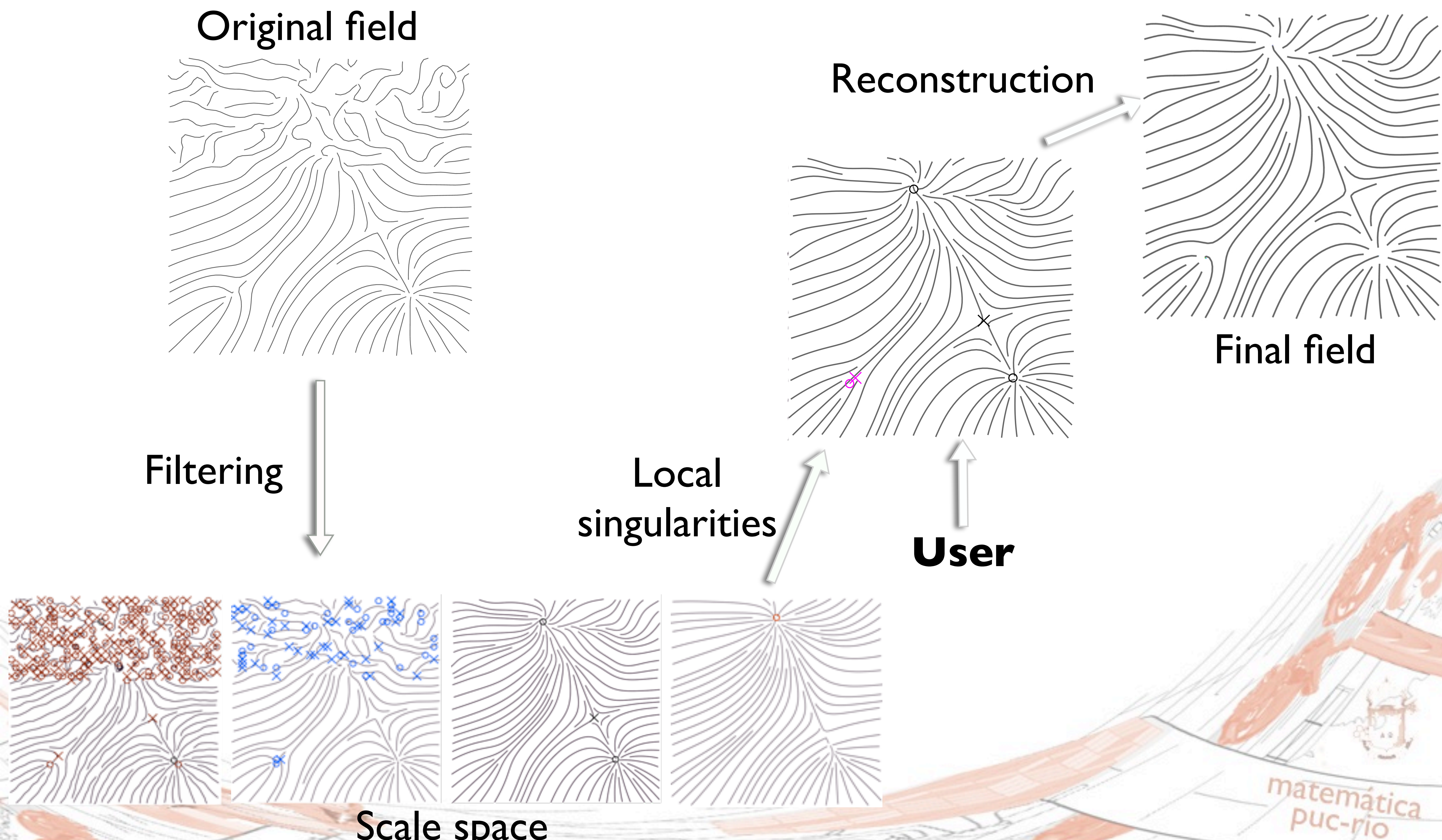
© Nascimento, Paixão, Lopes, L.

noise at the scale of the data

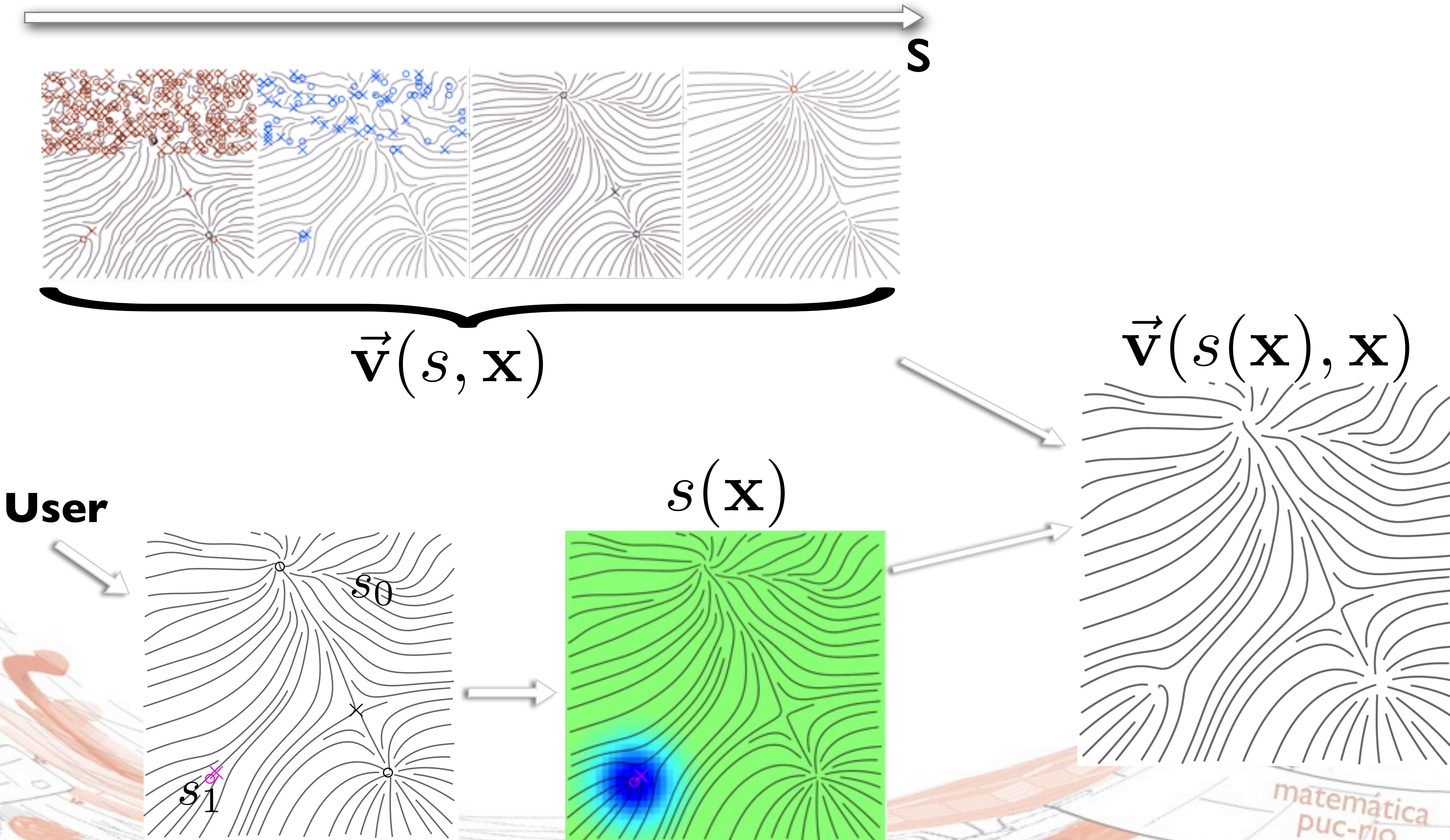
clean data + "important" vortices

local interpolation analysis

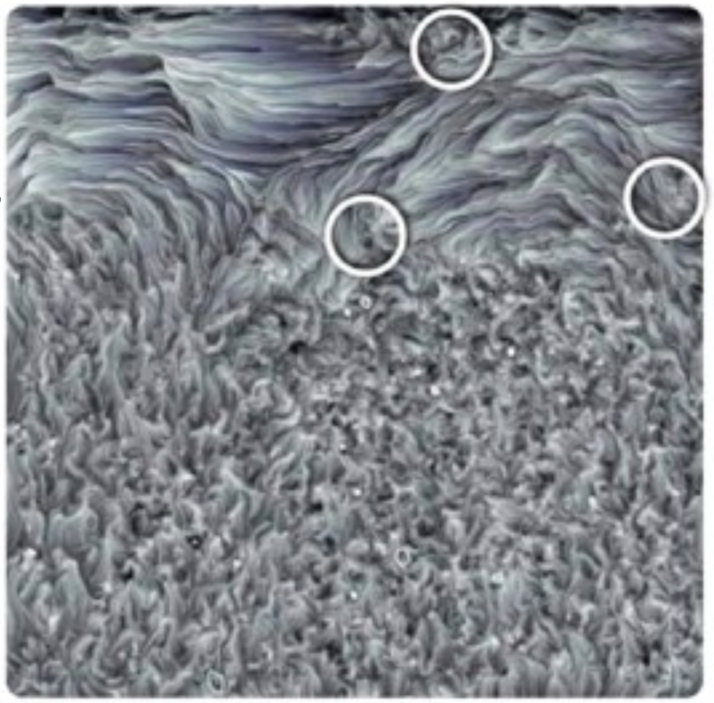
Interactive de-noising



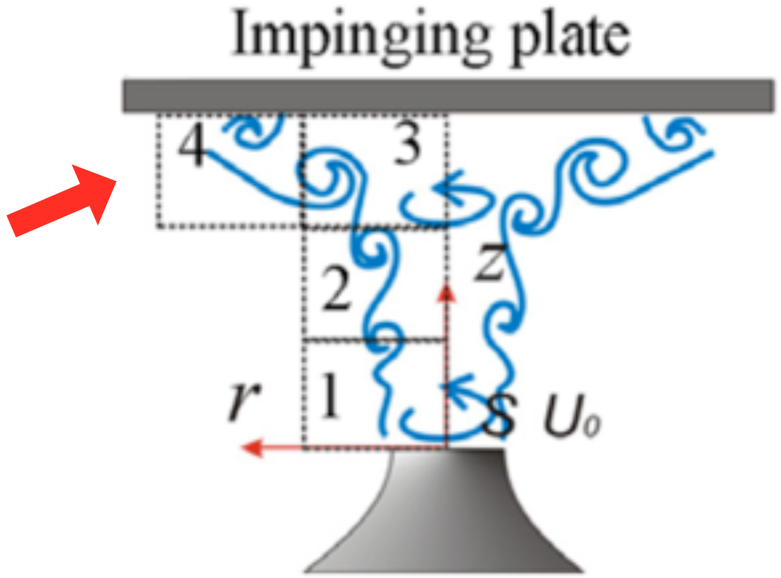
Scale-dependent singularity



Topology-aware de-noising



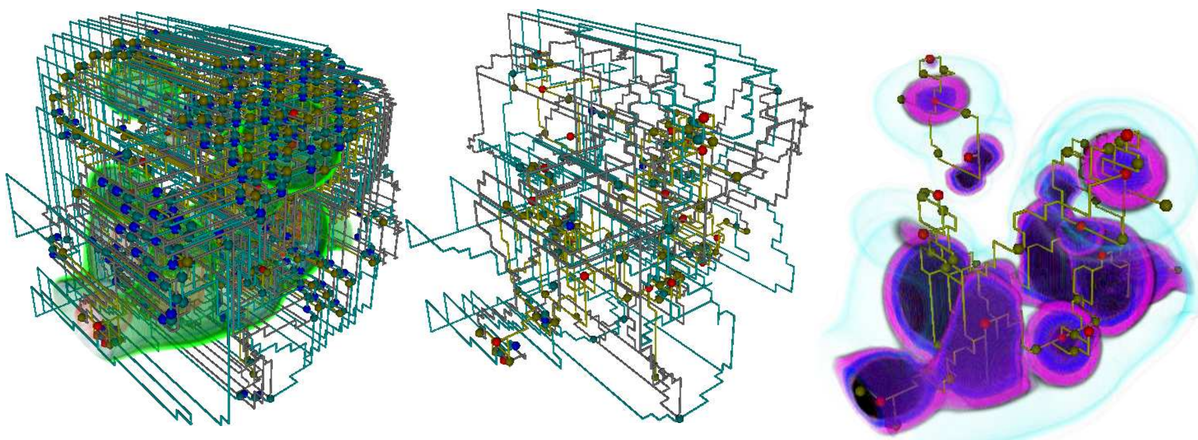
Original

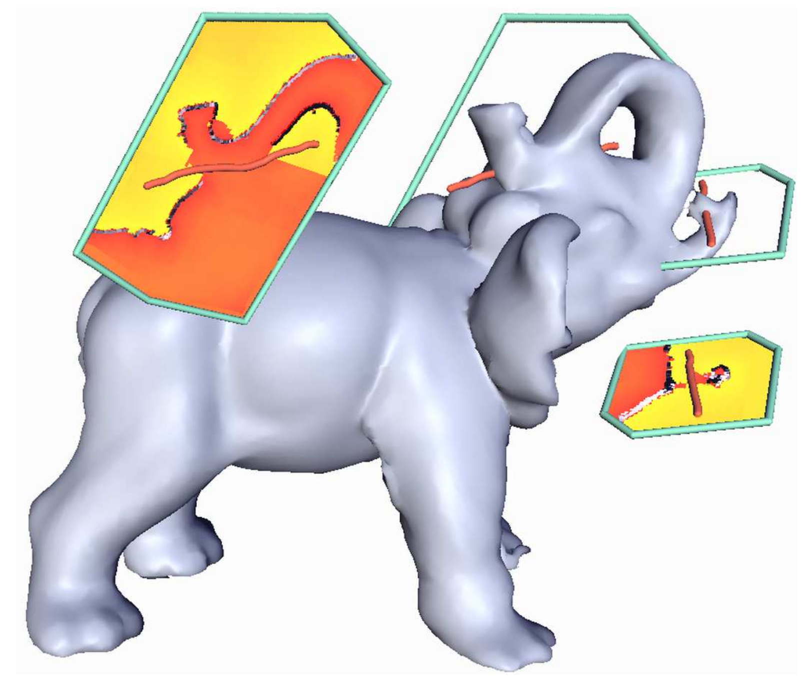


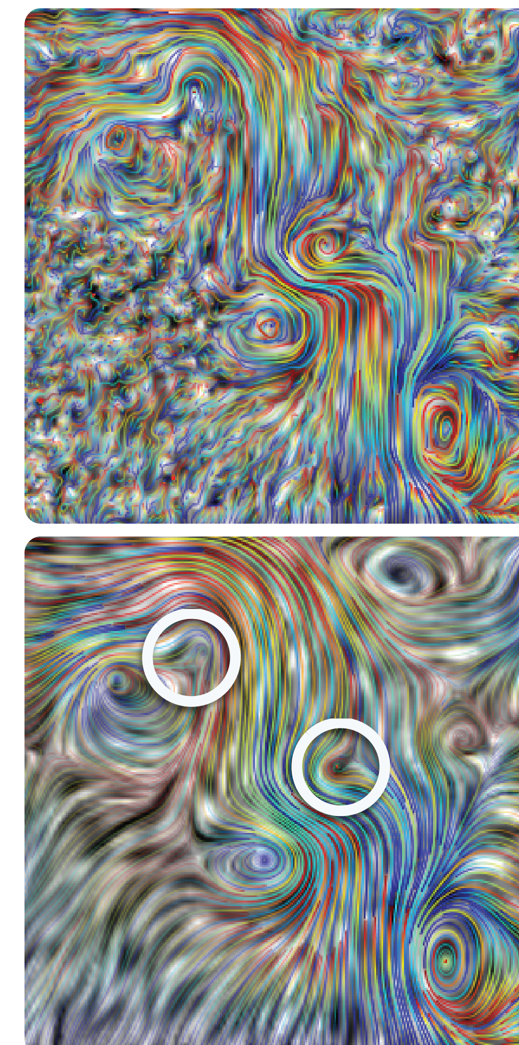
Smoothed

Reconstructed

Some common points







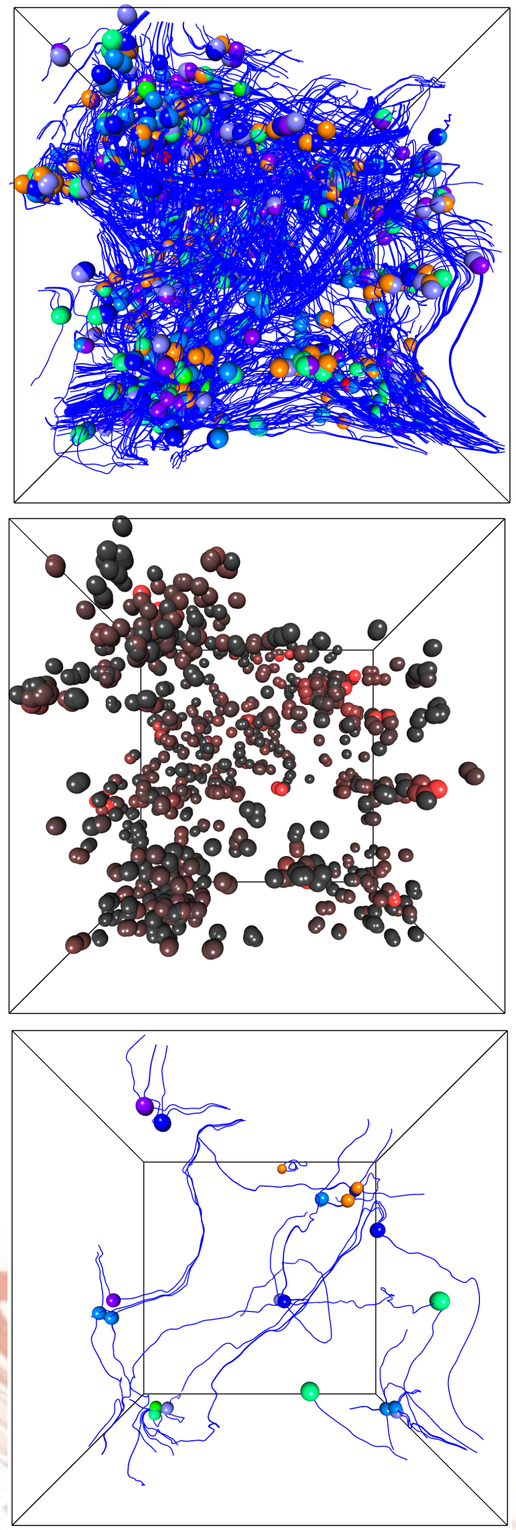
singular points only

$\Rrightarrow$ several applications

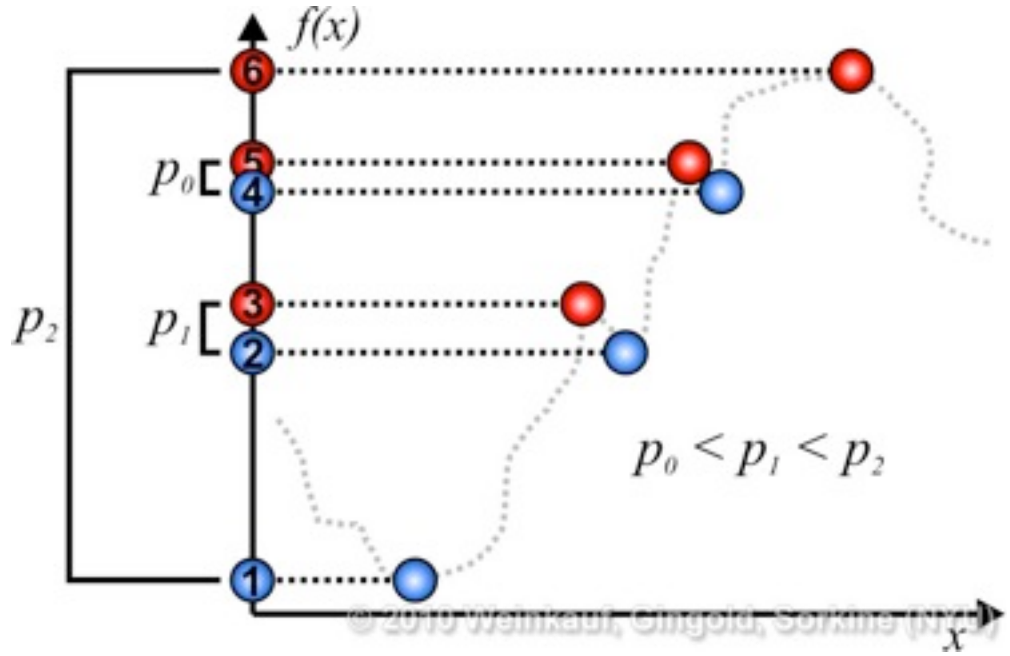
topology: intuitive interfaces

noise / scale problems

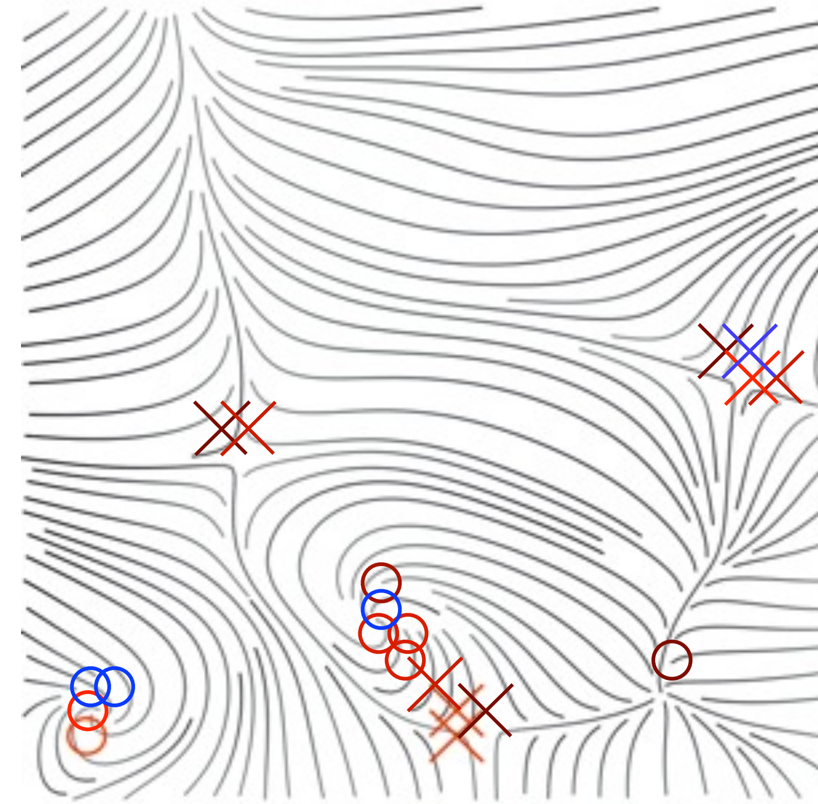
A note on noise / resolution



© Klein, Ertl



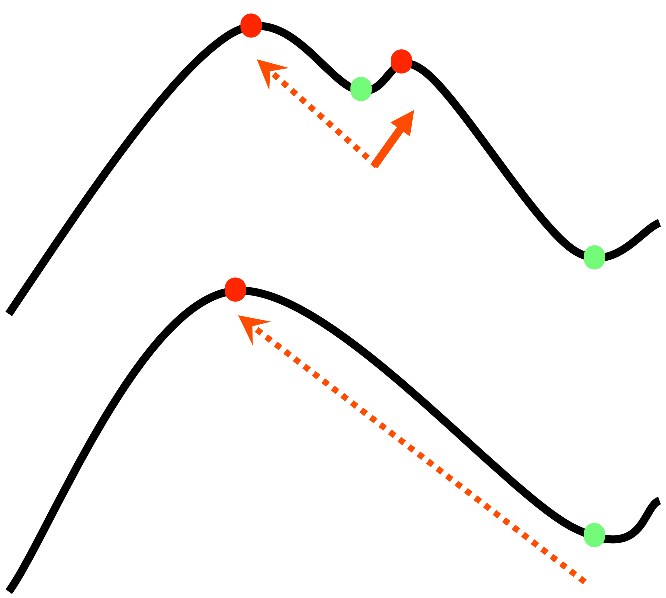
© Weinkauf, Gingold, Sorkine

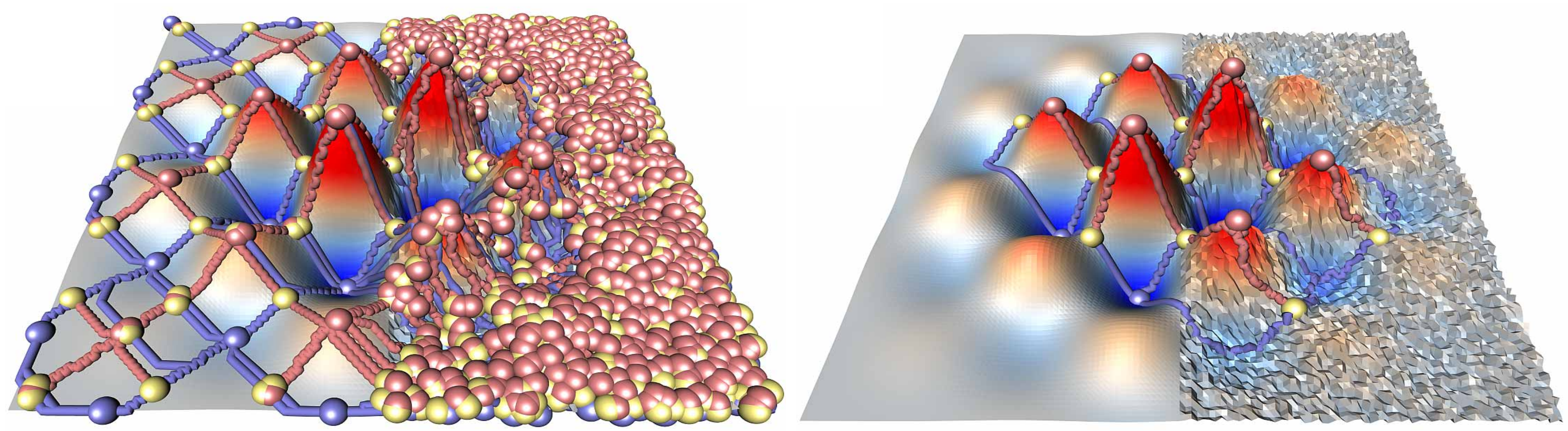


Persistence approach:

- Usual in Morse theory
- Smale used it for optimal Morse functions
- Keep singularities in place

Persistence is universal?





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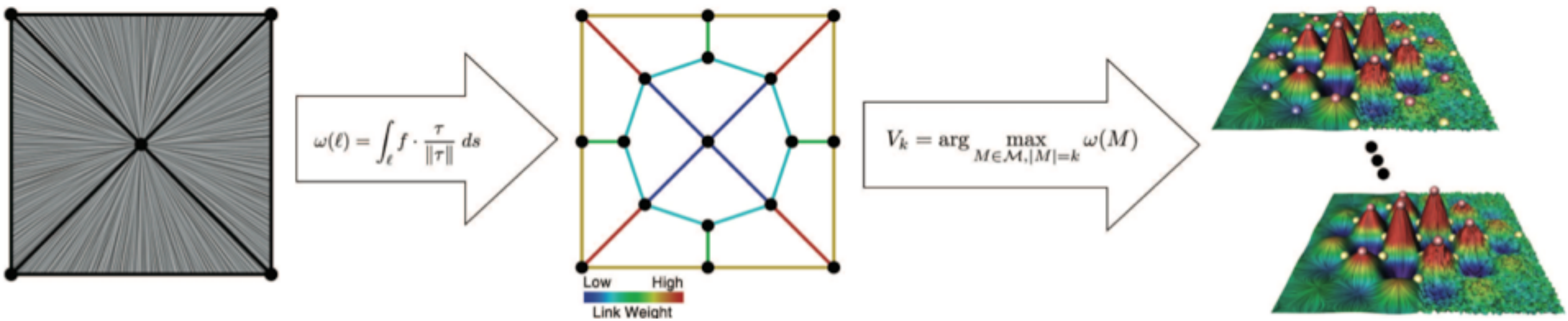
+ smooth and both discrete settings
- noise has a complex impact on topology
- discrete setting intrinsically resolution dependent on image and domain

$$f : \mathcal{M} \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

Scale-dependent critical set



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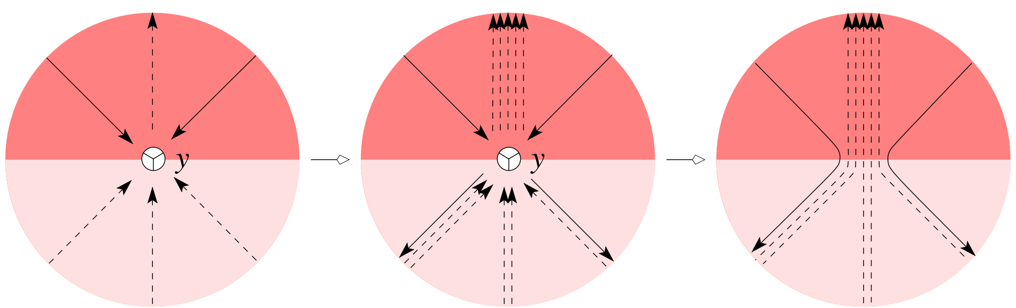
Forman’s critical set results from global construction:

number of critical cells

quality of the field approximation

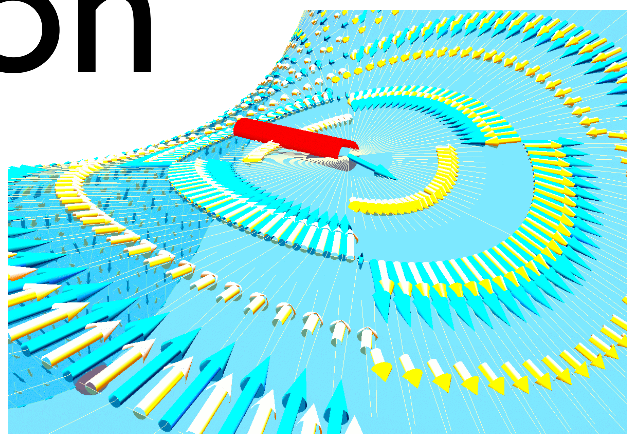
maximum weight matching

Partial conclusion



Banchoff

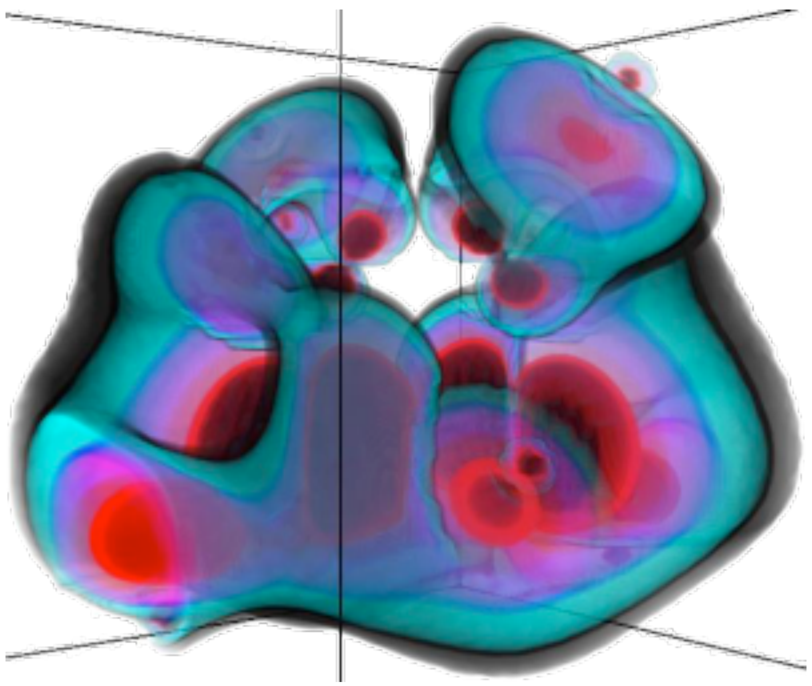
+ good critical points
+ intuitive!
- globally stable?



Forman

+ good global behavior
+ flexible discretization
- local precision?

Recent improvements



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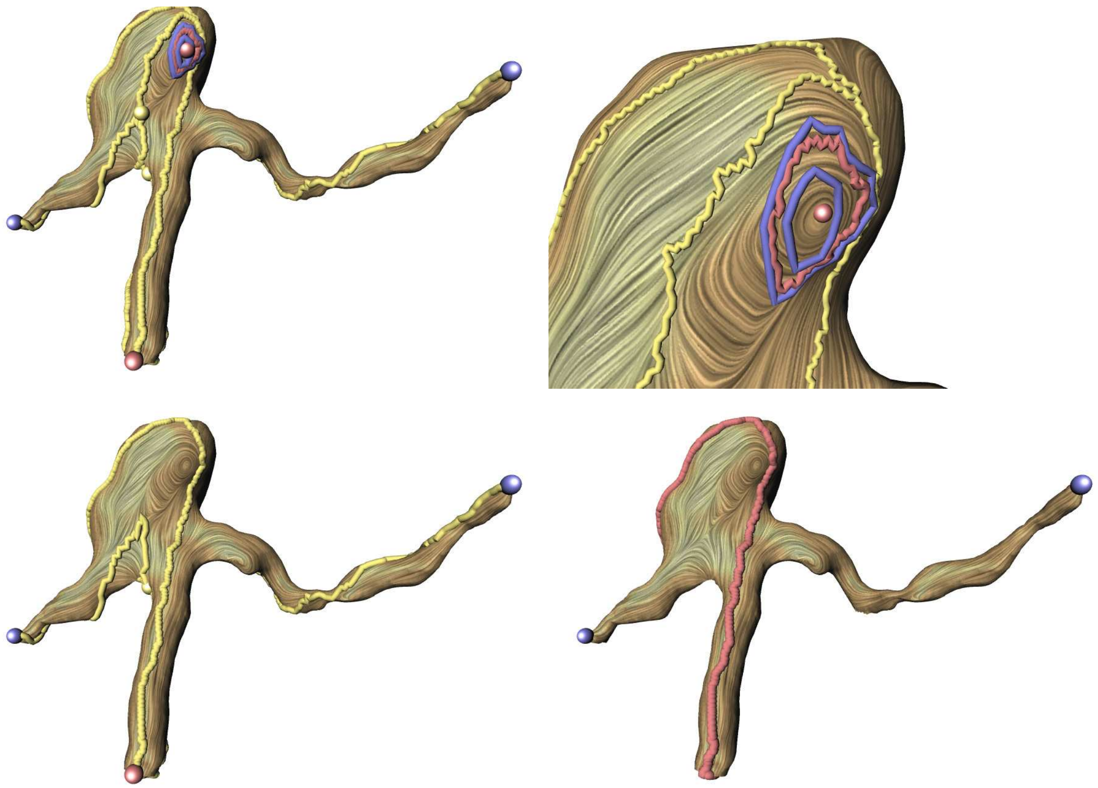

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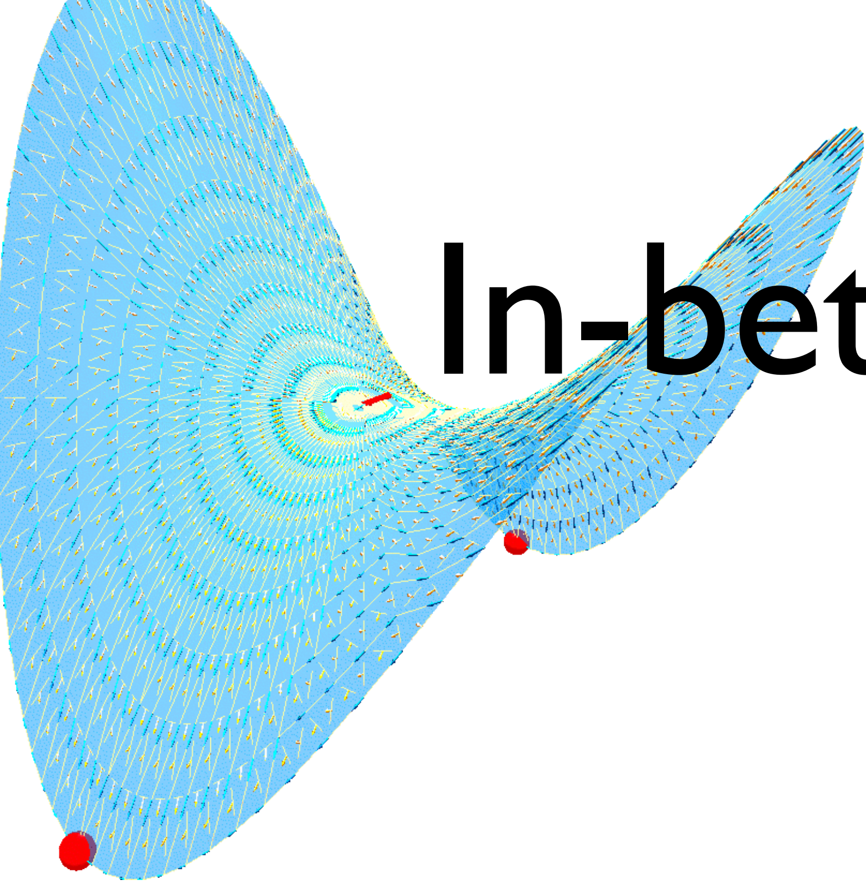
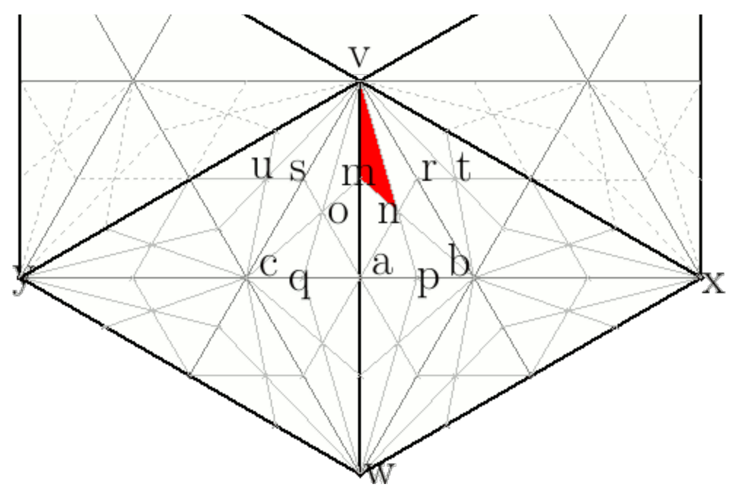
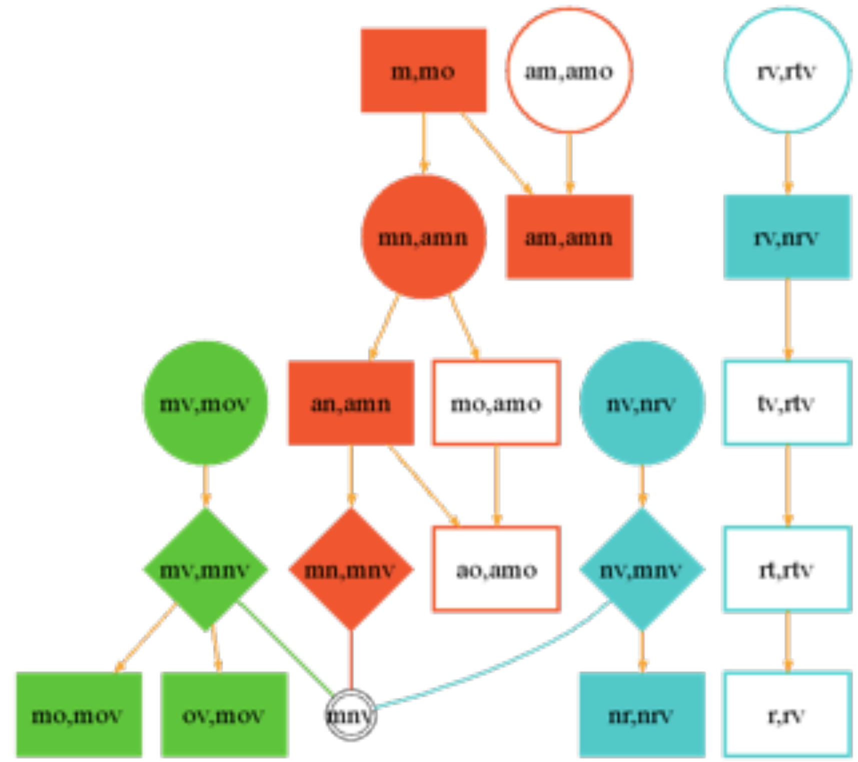
global stability of PL computations
(persistence and Morse-Smale)

control of critical cells in Forman's
(matching and localization)

In-between both worlds

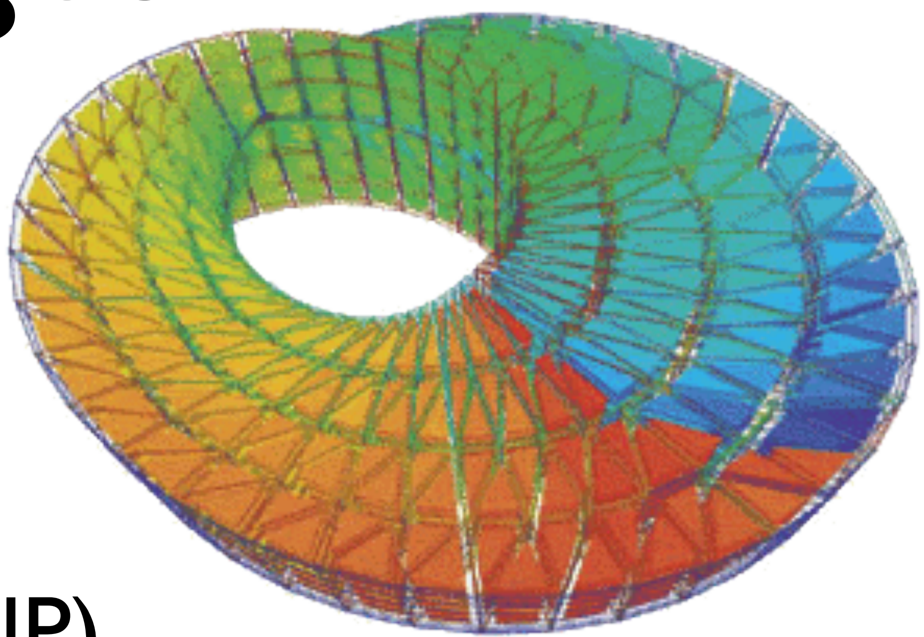






On triangulated surface, greedy construction of Forman's vector field keeps Banchoff's critical set for slowly varying function $f : \mathcal{K}_0 \hookrightarrow \mathbb{R}$

Next challenges



Higher dimension (besides NP)

L., Lopes, Tavares, Joswig, Pfetsch...

More general cases (infinite complexes)

Ayala, Vilches...

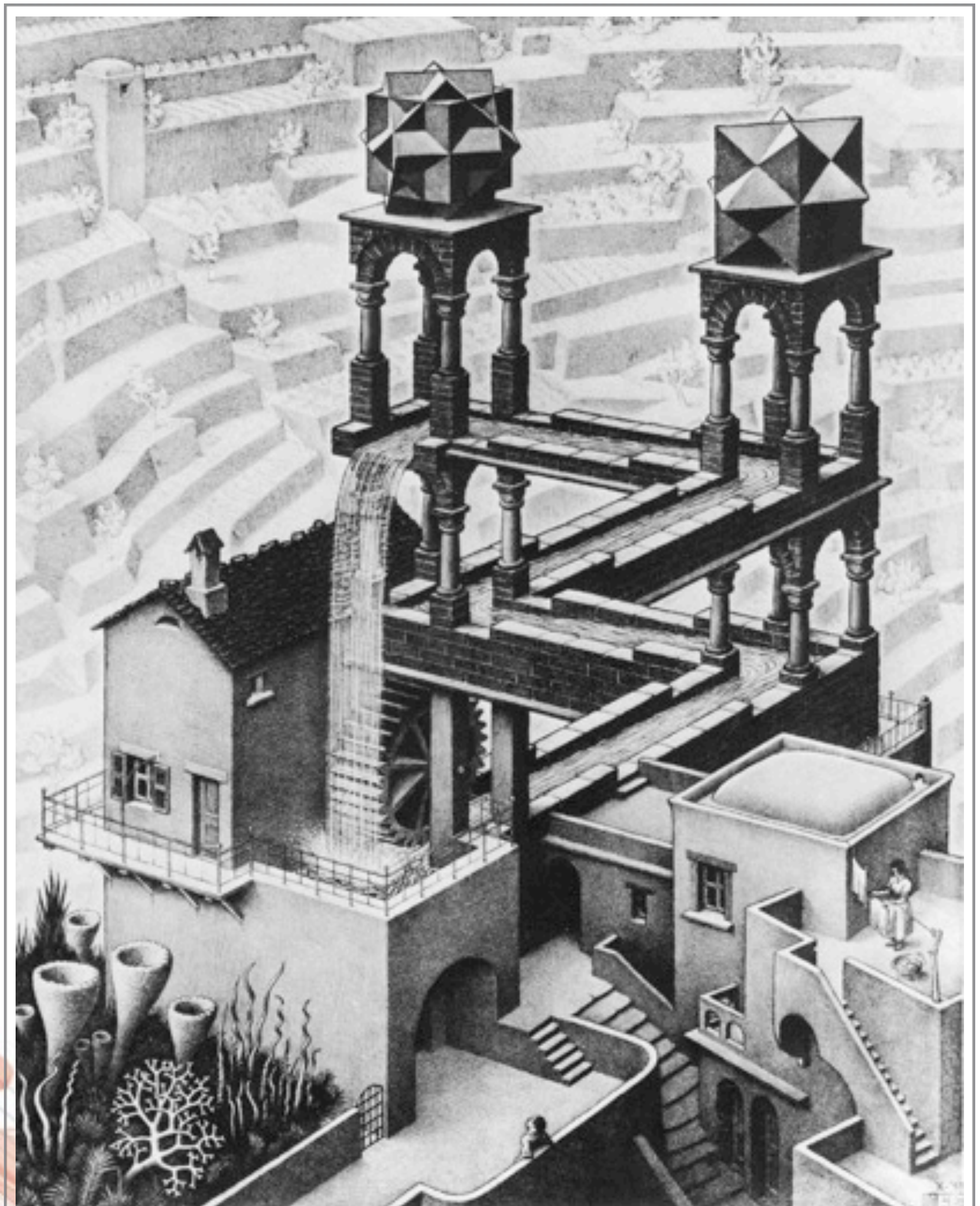
More complex objects (tensors, $\{f_i\}$)

Forman, Tricoche, Tong, Desbrun...

More theoretical guarantees

L., Zhang, Mischaikow...

Global from local?



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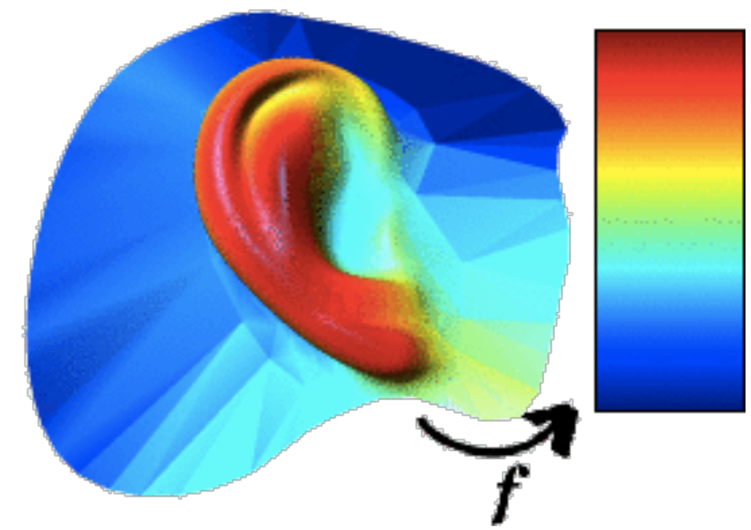

Thank you
for your attention!

Thomas Lewiner

PUC - Rio. Rio de Janeiro, Brazil!

http://thomas.lewiner.org/