

# Schnyder woods for higher genus triangulated surfaces

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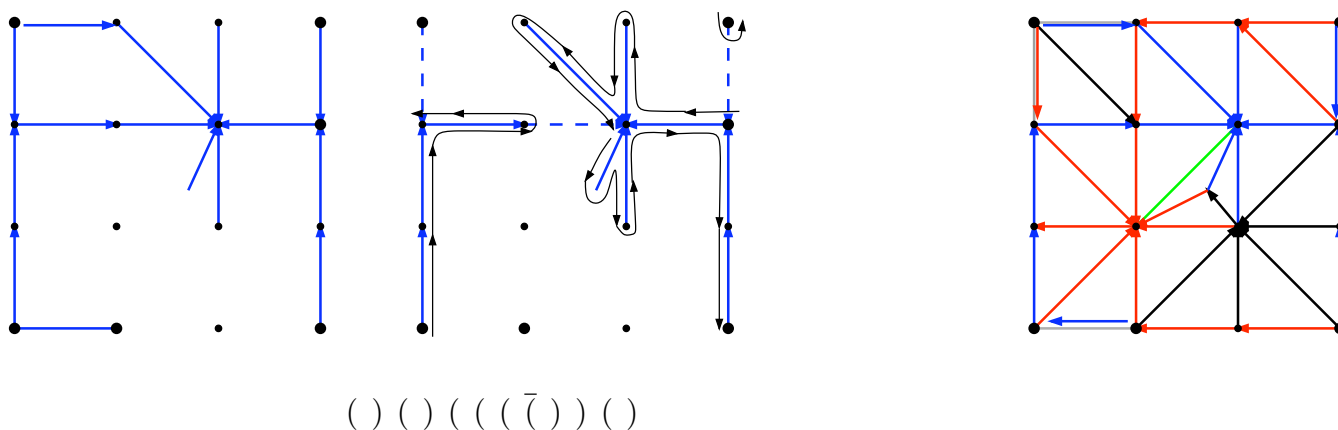
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**Abstract.** In this paper we introduce a unified framework for topological manipulation on triangulated 2-manifolds with or without boundary. We show that there are two kinds of primitive operators on the underlying meshes: operators which change the topological characteristic of the mesh and operators which just modify its combinatorial structure. We present such operators and demonstrate that they provide a complete and coherent set of elementary operations for mesh construction and edition.

**Keywords:** *Schnyder woods. Schnyder trees. Graph realizer. Handlebody theory. High genus surface. Topological graph theory.*



**Figure 1:** *Graph encoding on a torus using Schnyder woods.*

## 1 Introduction

Schnyder woods are a nice and deep combinatorial structure to finely capture the notion of planarity of a graph. They are named after W. Schnyder, who introduced these structures under the name of realizers and derived as main applications a new planarity criterion in terms of poset dimensions [29], as well as a very elegant and simple straight-line drawing algorithm [30]. There are several equivalent formulations of Schnyder woods, either in term of *angle labelling* (Schnyder labellings) or *edge colouring and orientation* or in terms of orientations with prescribed degrees. The most classical formulation is for the family of maximal plane graphs, i.e., plane triangulations, yielding the following striking property: the internal edges of a triangulation can be partitioned into three spanning trees rooted respectively at

each of the three vertices incident to the outer face. Schnyder woods, and more generally  $\alpha$ -orientations, received a great deal of attention [30, 14, 19, 17]. From the combinatorial point of view the set of Schnyder woods of a fixed triangulation has an interesting lattice structure [6, 2, 15, 12, 26], and the nice characterization in term of spanning trees motivated a large number of applications in several domains as graph drawing [30, 19], graph coding and sampling [11, 18, 3, 27, 16, 5, 9, 1]. Previous work focused mainly on the application and extension of the combinatorial properties of Schnyder woods to 3-connected plane graphs [14, 19]. In this article, we deal with combinatorial surfaces possibly having handles, i.e., oriented surfaces of arbitrary genus  $g \geq 0$ . Our main contribution is to show that it is possible to extend the local properties of Schnyder labelling in a coherent manner to triangulated surfaces. We investigate an important class of graphs, namely genus  $g$  triangulated surfaces, which correspond to the combinatorial structure underlying manifold triangle meshes of genus  $g$ .

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## (a) Related Work

### Vertex spanning tree decompositions

In the area of tree decompositions of graphs there exist some works dealing with the higher genus case. We mention one recent attempt to generalize Schnyder woods to the case of toroidal graphs [4] (genus 1 surfaces), based on a special planarizing procedure. In the genus 1 case it is actually possible to choose two adjacent non-trivial cycles, defining a so-called *tambourine*, whose removal makes the graph planar; the graph obtained can thus be endowed with a Schnyder wood. In the triangular case this approach yields a process for computing a partition of the edges into three edge-disjoint spanning trees plus at most 3 edges. Unfortunately, as pointed out by the authors the local properties of Schnyder woods are possibly not satisfied for a large number of vertices, because the size of the tambourine might be arbitrary large. Moreover, it is not clear how to generalize the method to genus  $g \geq 2$ .

### Planarizing surface graphs

A natural solution to deal with Schnyder woods (designed for planar graphs) in higher genus would consist in performing a planarization of the surface. Actually, given a surface  $S$  of genus  $g$  and size  $n$ , one can compute a cut-graph or a collection of  $2g$  non-trivial cycles, whose removal makes  $S$  a topological disk (possibly with boundaries). There is number of recent works [22, 33, 34, 13, 7, 21] dealing with interesting algorithmic challenges concerning the efficient computing of cut-graphs, optimal (canonical) polygonal schemas and shortest non-trivial cycles. For example there exist some works allowing to compute polygonal schemas in optimal  $O(gn)$  time for a triangulated orientable manifold [33, 22].

Nevertheless we point out that any strategy, based on such planarizing approach, would be not relevant for our purposes. From the combinatorial point of view this would imply to deal with boundaries of arbitrary size (arising from the planarizing procedure), as non-trivial cycles can be of size  $\Omega(\sqrt{n})$ , and cut-graphs have size  $O(gn)$ . Moreover, from the algorithmic complexity point of view, the most efficient procedures for computing small non-trivial cycles [7, 21] require more than linear time, the best known bound is currently of  $O(n \log n)$  time: this is a fundamental problem in the design of efficient linear time algorithms for graph encoding.

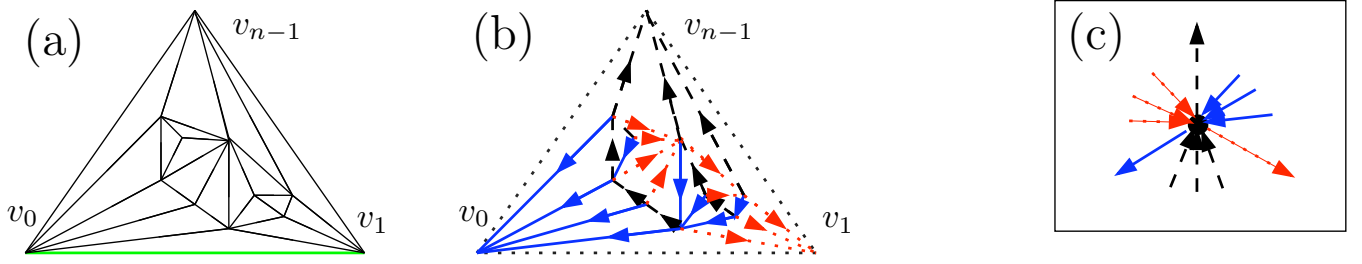
### Schnyder trees and graph encoding

One of our main motivations for generalizing Schnyder woods to higher genus is the great number of existing works in the domain of graph encoding and mesh compression that take advantage of spanning tree decompositions [31, 20, 28], and in particular of the ones underlying Schnyder woods (and related extensions) for planar graphs [11, 10, 18, 1, 27, 16]. The combinatorial properties of Schnyder woods and the related characterizations (*canonical orderings* [19]) for planar graphs yield efficient procedures for encoding tree

structures based on multiple parenthesis words. In this context a number of works have been proposed for the simple compression [18] or the succinct encoding [11, 10] of several classes of planar graphs. More recently, this approach based on spanning tree decompositions has been further extended to design a new succinct encoding of labelled planar graphs [1]. Once again, the main ingredient is the definition of three traversal orders on the vertices of a triangulation, directly based on the properties of Schnyder woods. Finally we point out that existence of *minimal orientations* (orientations without counter-clockwise directed cycles) recently made it possible to design the first optimal (linear time) encodings for some popular classes of planar graphs. For the case of triangulations and 3-connected plane graphs [27, 16] there exist some bijective correspondences between such graphs and some special classes of plane trees, which give nice combinatorial interpretations of enumerative formulas originally found by Tutte [32]. Very few works have been proposed for dealing with higher genus surface graphs (corresponding to manifold meshes): this is due to the strong combinatorial properties involved in the planar case. Nevertheless, the topological approach used by *Edgebreaker* (using at most 3.67 bits per vertex in the planar case) has been successfully adapted to deal with triangulated surfaces having arbitrary topology: orientable manifold with handles [24] and also multiple boundaries [23]. Using a different approach, based on a partitioning scheme and a multi-level hierarchical representation [8], it is also possible to obtain an encoding of a triangulation of fixed genus  $g$ , having  $f$  faces and  $n$  vertices, requiring  $2.175f + O(g \log f) + o(f)$  bits (or  $4.35n + o(gn)$  bits) which is optimal for surfaces with a boundary: nevertheless, the amount of additional bits hidden in the sub-linear  $o(n)$  term can be quite large, of order  $\Theta(\frac{n}{\log n} \log \log n)$ .

## (b) Contributions

Our first result consists in defining new traversal orders of the vertices of a triangulation of genus  $g$ , as extension of the *canonical orderings* defined for planar graphs. We are also able to provide a generalization of the Schnyder labelling to the case of higher genus surfaces. The major novelty is in the way we show that the linear time algorithm designed for the planar case can be extended in a nontrivial way in order to design a traversal of a genus  $g$  surface. This induces a special edge colouring and orientation that is a natural generalization of the corresponding planar structure. In particular, the spanning property characterizing Schnyder wood is again verified almost everywhere in the genus  $g$  case. Our approach involves implicitly the computing of cut-graphs, as one would expect. Nevertheless, our strategy is really different from the one based on planarizing procedures: even if our cut-graphs and non-trivial cycles can have arbitrary size, we show that it is always possible to propagate the spanning condition of Schnyder woods even along this cuts in a coherent and natural manner. This property, and in particular the fact the there are only few exceptions to the spanning con-



**Figure 2:** (a) A rooted planar triangulation, (b) endowed with a Schnyder wood (b). (c) The local condition of Schnyder woods.

dition, allow us to characterize our graph decomposition in term of one-face maps of genus  $g$  (a natural generalization of plane trees). We think that this work is a first step in the attempt of generalizing the properties of Schnyder woods and more generally  $\alpha$ -orientations (orientations with prescribed out-degrees) to other interesting classes of graphs with no spherical topology.

## 2 Preliminaries

### (a) Graphs on surfaces.

In this work we consider *discrete surfaces* such as triangular meshes, and we consider only the topological information specific to such a surface, i.e., the incidences vertices/edges/faces. All discrete surfaces we consider can be precisely formalised with the concept of map. A *map* is a graph embedded on a surface  $\mathcal{S}$  such that the areas delimited by the graph, called the *faces* of the map, are topological disks. In the following the maps considered have neither loop nor multiple edges, which means that the embedded graphs are simple. Of particular for us are maps of genus  $g$  with all faces of degree 3, we call such a map a *triangulated surfaces* of genus  $g$ . These are the combinatorial part of triangular meshes of a surface of genus  $g$ . Given a surface  $\mathcal{S}$ , we refer to its *primal graph* (or *1-skeleton*) as the underlying graph consisting of its edges and vertices. The *dual graph* of  $\mathcal{S}$  is defined as the graph having the faces as nodes and whose edges correspond to pairs of adjacent faces in  $\mathcal{S}$ . Given a map  $M$  on a surface  $\mathcal{S}$ , a *cut graph* of  $M$  is a subgraph of  $M$  whose induced embedding on  $\mathcal{S}$  defines a one-face map. If  $\mathcal{S}$  is cut along such a submap  $G^c$ , the surface obtained is equivalent to a topological disk; the boundary of the disk contains each edge of  $G^c$  twice, and the original surface can be obtained by identifying and gluing together these pairs of edges, which defines a so-called *polygonal scheme* of the surface  $\mathcal{S}$ . For a detailed presentation of the topological properties of graphs on surfaces we refer to [25].

### (b) Schnyder woods for Plane Triangulations

Let us first recall the definition of Schnyder woods for plane triangulations, which we will later generalize to higher genus. While the definition is given in terms of local conditions, the main structural property is more global, namely a partition of the edges into 3 (essentially) spanning trees.

**Definition 1 ([30])** Let  $T$  be a plane triangulation, and denote by  $v_0, v_1, v_{n-1}$  the vertices of the outer face in cw order; we denote by  $\mathcal{E}$  the set of edges of  $T$  except those three incident to the outer face. A **Schnyder wood** of  $T$  is an orientation and labelling, with label in  $\{0, 1, 2\}$  of the edges in  $\mathcal{E}$  so as to satisfy the following conditions:

- **root face condition:** the edges incident to the vertices  $v_0, v_1, v_{n-1}$  are all ingoing and are respectively of colour 0, 1, and 2.
- **local condition:** For each vertex  $v$  not incident to the root face, the edges incident to  $v$  in ccw order are: one outgoing edge coloured 0, zero or more incoming edges coloured 2, one outgoing edge coloured 1, zero or more incoming edges coloured 0, one outgoing edge coloured 2, and zero or more incoming edges coloured 1, which we write concisely as:  $(\text{Seq}(\text{In}1), \text{Out}0, \text{Seq}(\text{In}2), \text{Out}1, \text{Seq}(\text{In}0), \text{Out}2)$ .

A fundamental property of Schnyder woods [30] is the following

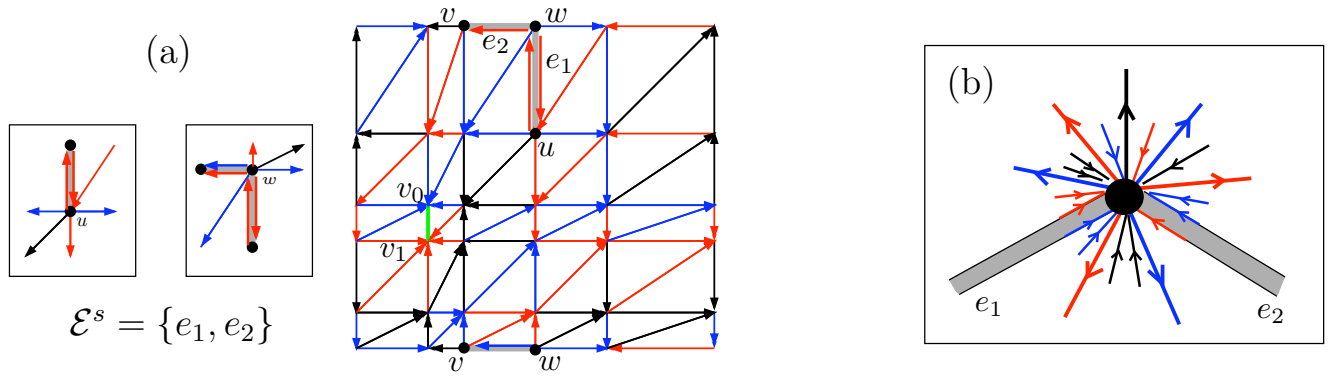
**Fact 1** Each plane triangulation  $T$  admits a Schnyder wood. Given a Schnyder wood on  $T$ , the three oriented graphs  $T_0, T_1, T_2$  induced by the edges of colour 0, 1, 2 are trees that span all inner vertices and are naturally rooted at  $v_0, v_1$ , and  $v_{n-1}$ , respectively.

## 3 Higher genus triangulations

### (a) Generalized Schnyder Woods

As done in previous works considering the extension of Schnyder woods to other classes of (not triangulated) planar graphs [14, 27], one main contribution is to propose a new generalized version of Schnyder woods to genus  $g$  triangulations (see Figure 3 for an example).

**Definition 2** Consider a triangulated surface  $\mathcal{S}$  of genus  $g$ , having  $n$  vertices and a root face  $(v_0, v_1, v_{n-1})$ ; let  $\mathcal{E}$  be the set of edges of  $\mathcal{S}$  except those three incident to the root face. A genus- $g$  Schnyder wood on the surface  $\mathcal{S}$  is a partition of  $\mathcal{E}$  into a set of normal edges and a set  $\mathcal{E}^s = \{e_1, e_2, \dots, e_{2g}\}$  of  $2g$  special edges considered as *fat*, i.e., made of two parallel edges. In addition, each edge,



**Figure 3:** (a) A face-rooted triangulated surface  $\mathcal{S}$  of genus 1 endowed with a  $g$ -Schnyder wood. (b) The local condition for vertices incident to special edges.

a normal edge or one of the two edges of a special edge, is simply oriented and has a label in  $\{0, 1, 2\}$ , so as to satisfy the following conditions:

- **root face condition:** All edges incident to  $v_0, v_1,$  and  $v_{n-1}$  are ingoing with colour 0, 1, and 2, respectively.
- **local condition for vertices not incident to special edges:** for every vertex  $v \in \mathcal{S} \setminus (\{v_0, v_1, v_{n-1}\})$  not incident to any special edge, the edges incident to  $v$  in ccw order are, as in the planar case, of the form:

(Seq(In1), Out0, Seq(In2), Out1, Seq(In0), Out2).

- **local condition for vertices incident to special edges:** A vertex  $v$  incident to  $k \geq 1$  special edges has exactly one outgoing edge in colour 2. Consider the  $k + 1$  sectors around  $v$  delimited by the  $k$  special edges and the outgoing edge in colour 2. Then in each sector, the edges occur as follows in cw order:

Seq(In1), Out0, Seq(In2), Out1, Seq(In0).

The edge-based characterization of a triangulated surface proposed above has some advantages that make it a natural generalization of the planar definition. As one would expect, the planar and the genus  $g$  definition do coincide when considering planar triangulations. Second we observe that the local condition is again true almost everywhere (except the few vertices lying in  $\mathcal{E}^s$ ): in particular almost all the vertices have out-degree 3, which makes the  $g$ -Schnyder woods a good characterization of the local planarity of a bounded genus surface. Finally, we point out that the local condition above leads to graph decompositions expressed in term of one-face maps having genus  $g$ , which are the natural generalization of plane spanning trees (see Theorem 8 and the remark after).

### (b) Handle operators: notations and definitions

Following the approach suggested in [24], based on Handlebody theory for surfaces, we design new traversal strategy for higher genus surfaces: as in the planar case, our strategy consists in conquering the whole graph incrementally, face by face, using a vertex-based operator (`conquer`) and two new operators (`split` and `merge`) designed to represent the handle attachments. We start by setting some notations and definitions. We consider a triangulated surface  $\mathcal{S}$  having genus  $g$  and  $n$  vertices. We denote by  $S^{out}$  ( $S^{in}$ ) the embedded subgraph of  $\mathcal{S}$  induced by the face already conquered (not yet conquered, respectively).  $S^{out}$  is a face-connected map of genus  $g$  having  $b \geq 1$  boundaries, each boundary being a simple cycle  $C_i, i \in [1..b]$ . We define  $\partial S^{in} := \cup_{i=1}^b C_i$  as the overall border between  $S^{in}$  and  $S^{out}$ .

#### Handle operator of first type.

**Definition 3** A chordal edge is an edge of  $S^{out} \setminus \partial S^{in}$  whose two extremities are on  $\partial S^{in}$ . A boundary vertex  $w \in C_i$  is free if  $w$  is not incident to a chordal edge  $e$ .

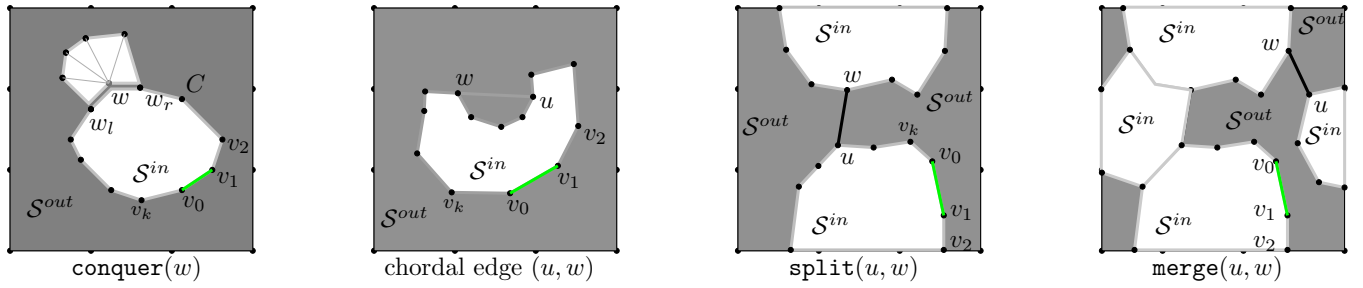
We can now introduce the first operator, called `conquer`, which adds a free vertex  $w$  with all its incident edges and faces to  $S^{in}$ . More precisely, the *conquest* or *removal* of a vertex  $w$  consists in attaching a set of triangles to  $S^{in}$  as follows (see the first picture in Figure 4):

- `conquer`( $w$ ), with  $w$  a free vertex: Update the conquered area, by transferring from  $S^{out}$  to  $S^{in}$  all faces incident to  $w$  that were not yet in  $S^{in}$ . The vertices and edges of  $S^{out}$  ( $S^{in}$ ) are naturally updated as those vertices and edges incident to the faces of  $S^{out}$  ( $S^{in}$ , respectively).

This operation corresponds to a handle operator of type 1 (see [24]), not modifying the topology of  $S^{in}$ .

#### Handle operators of second type.

Consider a triangulated surface  $\mathcal{S}$  of genus  $g$  and a collection of  $b$  cycles  $\{C_i\}_i$  delimiting a face-connected map  $S^{out}$ . A chordal edge  $e$  for  $S^{out}$  is said to be *nonseparating* if  $S^{out}$  is not disconnected when cutting along  $e$ . A chordal



**Figure 4:** Illustrated on a toroidal graph, from left to right, a conquer operation, a contractible chordal edge, a split edge, and a merge edge.

edge whose two extremities are in the same cycle  $C_i$  is called a *splitting chordal edge*. For such an edge, we define  $C'$  and  $C''$  as the two cycles formed by  $C_i + e$ . Then  $e$  is said to be *contractible* if either  $C'$  or  $C''$  is contractible

**Definition 4 (split edge)** A split edge for the area  $S^{out}$  is a nonseparating splitting chordal edge.

It is easy to see that the removal of a split edge  $e$  produces a face-connected map  $S^{out} \setminus \{e\}$  with  $b+1$  boundaries, hence having Euler characteristic  $\chi' = 2 - 2g - (b+1) = \chi - 1$ .

**Definition 5 (merge edge)** A merge edge for the area  $S^{out}$  is a nonseparating chordal edge  $e$  having its two extremities on two distinct cycles  $C_i$  and  $C_j$ ,  $i \neq j$ .

This time the removal of the edge  $e$  produces a face-connected map  $S^{out} \setminus e$  with  $b-1$  boundaries, hence having Euler characteristic  $\chi' = 2 - 2g - (b-1) = \chi + 1$ .

We are now able to define two new operators to perform handle attachments. The `split` and `merge` operators defined below are (the inverse of) handle operators of type 1: they are designed to identify boundary vertices lying on distinct cycles. Intuitively, the `split` operator *splits* a boundary curve into two distinct components, thereby increasing the number of boundaries of  $\partial S^{in}$  by 1; while a *merge* operator merges two boundaries, thereby decreasing the the number of boundaries of  $\partial S^{in}$  by 1.

Let  $\mathcal{S}$  be a triangulated surface of genus  $g$  and let  $C_i, i \in [1..b]$  be a collection of  $b$  cycles delimiting a face-connected map  $S^{out}$ . Given a split edge  $e = (u, w)$  having its extremities on a cycle  $C$ , the operator `split`( $e$ ) produces the splitting of  $C = \{v_0, v_1, \dots, v_k\}$  (with  $u = v_{i_1}$  and  $w = v_{i_2}$ , for some indices  $i_1 < i_2 \leq k$ ) into two new cycles  $C', C''$ . More precisely  $C'$  is the new simple cycle defined by the set of vertices  $\{v_0, v_1, \dots, v'_{i_1}, v'_{i_2}, v_{i_2+1}, \dots, v_{k-1}, v_k\}$ , where  $v'_{i_1}$  and  $v'_{i_2}$  are copies of the vertices  $v_{i_1}$  and  $v_{i_2}$ . In similar way, a new cycle  $C''$  is created by considering two duplicated vertices  $v''_{i_1}$  and  $v_{i_2}$  (copies of  $v_{i_1}$  and  $v_{i_2}$ ) and the list of vertices  $\{v''_{i_1}, v''_{i_2}, v_{i_2+1}, \dots, v_{i_1-1}\}$ .

A merge operation can be performed in a similar way, on a given merge edge  $e = (v'_{i_1}, v''_{i_2})$  whose extremities belong to different boundaries  $C'$  and  $C''$ ,

and produces a new simple cycle  $\bar{C}$  containing two copies of  $v'_{i_1}$  and  $v''_{i_2}$  and all the vertices in  $C'$  and  $C''$ . More precisely, denoting by  $\{v'_0, \dots, v'_{k'}\}$  the set of vertices defining  $C'$  and  $\{v''_0, \dots, v''_{k''}\}$  the vertices of  $C''$ , the new simple cycle  $\bar{C}$  is defined by  $\{v'_0, \dots, v'_{i_2}, v''_{i_2}, v''_{i_2+1}, \dots, v''_{k''}, v'_0, \dots, v'_{i_2}, v'_{i_1}, v'_{i_1+1}, \dots, v'_{k'}\}$

The edges involved in the merge/split operations are respectively called merge edges and split edges. These will be exactly the special edges of the genus  $g$  Schnyder wood to be computed by our traversal algorithm. The vertices incident to merge or split edges are called *multiple vertices*, as each merge/split operation results in considering such a vertex as duplicated.

### (c) Computing a Schnyder wood of a planar triangulation

In this section we briefly review a well known linear time algorithm designed for computing a Schnyder wood of a planar triangulation, following the presentation by Brehm [6]. Once defined the `conquer` operation, we can associate to it a simple rule for colouring and orienting the edges incident to a vertex conquered:

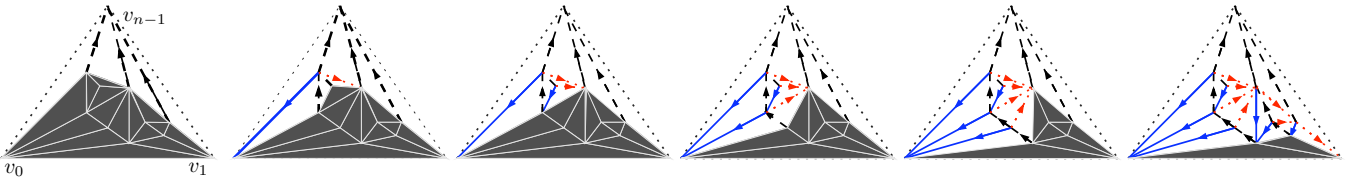
- `colorient`( $w$ ): orient outward of  $v$  the two edges connecting  $v$  to its two neighbours on  $\partial S^{in}$ ; assign color 0 (1) to the edge connected to the left (right, respectively) neighbour, looking toward  $S^{out}$ . Orient toward  $v$  and color 2 all edges incident to  $v$  in  $S^{out} \setminus \partial S^{in}$ .

We can now formulate the algorithm for computing a Schnyder wood as a sequence of  $n-2$  `conquer` and `colorient` operations, following a simple and elegant presentation adopted by Brehm [6]. Given a plane triangulation  $\mathcal{T}$  with outer face  $\{v_0, v_1, v_{n-1}\}$  we start with  $S^{in}$ —the set of visited vertices—initialized to  $\{v_0, v_1, v_{n-1}\}$ , and with  $\partial S^{in}$ —the border of  $S^{in}$ —initialized to be the contour of the outer face.

`COMPUTESCHNYDERPLANAR`( $\mathcal{T}$ ) ( $\mathcal{T}$  a plane triangulation)

```
while  $C \neq \{v_0, v_1\}$ 
  Choose a free vertex  $v$  on  $C$ ;
  conquer( $v$ ); colorient( $v$ );
end while
```





**Figure 5:** Some steps of the algorithm that computes a Schnyder wood of a planar triangulation  $\mathcal{T}$  (rooted at  $(v_0, v_1)$ ). The traversal is performed as a sequence of conquer operations on free vertices: edges in  $S^{out}$  (dark region) are oriented and directed according to the Schnyder rule.

The correctness and termination of the traversal algorithm above is based on the fundamental property that a triangulated topological disk always has on its boundary a vertex not incident to any internal chord.

One major advantage of computing a Schnyder wood in an incremental way is that we are able to put in evidence some invariants, which remain satisfied at each step of the algorithm:

- for each vertex  $v \in S^{in}$ , all edges incident to  $v$  are coloured and directed in such a way that the Schnyder rule is always satisfied;
- no edge in  $S^{out}$  (the not visited region) is coloured or oriented;
- every boundary node  $v \in C \setminus \{v_0, v_1\}$  has exactly one outgoing edge lying in  $S^{in}$  and having colour 2. All other possibly existing edges lying in  $S^{in}$  which are incident to  $v$  verify the Schnyder local condition (see local condition in Definition 1).

Using these invariants that are illustrated in the two first pictures of Figure 7, it is straightforward to prove the following lemma (see [6] for a detailed presentation):

**Lemma 6** *Given a planar triangulation  $\mathcal{T}$  with outer face  $(v_0, v_1, v_{n-1})$  the algorithm COMPUTESCHNYDERPLANAR computes in linear time a Schnyder wood of  $\mathcal{T}$ .*

#### (d) A new traversal algorithm for genus $g$ surfaces

This section presents our main contribution, an algorithm for traversing a triangulated surface  $S$  of arbitrary genus  $g \geq 0$ , which extends the algorithm of Brehm in a natural way. As in the planar case, the traversal is a greedy sequence of conquer operations on free vertices, with here the important difference that these operations are interleaved with  $2g$  merge/split operations so as to make the genus of the conquered area increase from 0 to  $g$ . We start with  $\mathcal{C} := \{C_0\} := \{(v_0, v_1, v_{n-1})\}$ . At the beginning  $S^{in}$  is a topological disk delimited by the simple cycle  $C_0$ . As in the planar case, we make use of the operation `conquer`( $v$ ) that consists in transferring the (remaining) faces incident to  $v$  from  $S^{out}$  to  $S^{in}$ . Associated with such a conquest is the `colorient` rule for coloring and orienting the edges conquered, as defined in Section 3(c). We also make use of the handle operations `split` and `merge`, as defined in Section 3(b).

COMPUTESCHNYDERANYGENUS( $S$ ) ( $S$  a triangulated surface of genus  $g$ )

while  $\{\mathcal{C}\} \neq \{v_0, v_1\}$

  If there exists a free vertex  $v$  on some  $C_i \in \mathcal{C}$

`conquer`( $v$ );   `colorient`( $v$ );

    otherwise, if there exists a split edge  $e = (u, w) \in S^{out}$ ;

`split`( $u, w$ ); thereby adding a new cycle  $C'$  to  $\mathcal{C}$ ;

    otherwise, find a merge edge  $e = (u, w) \in S^{out}$ ;

`merge`( $u, w$ ); thereby removing a cycle from  $\mathcal{C}$ ;

  end while

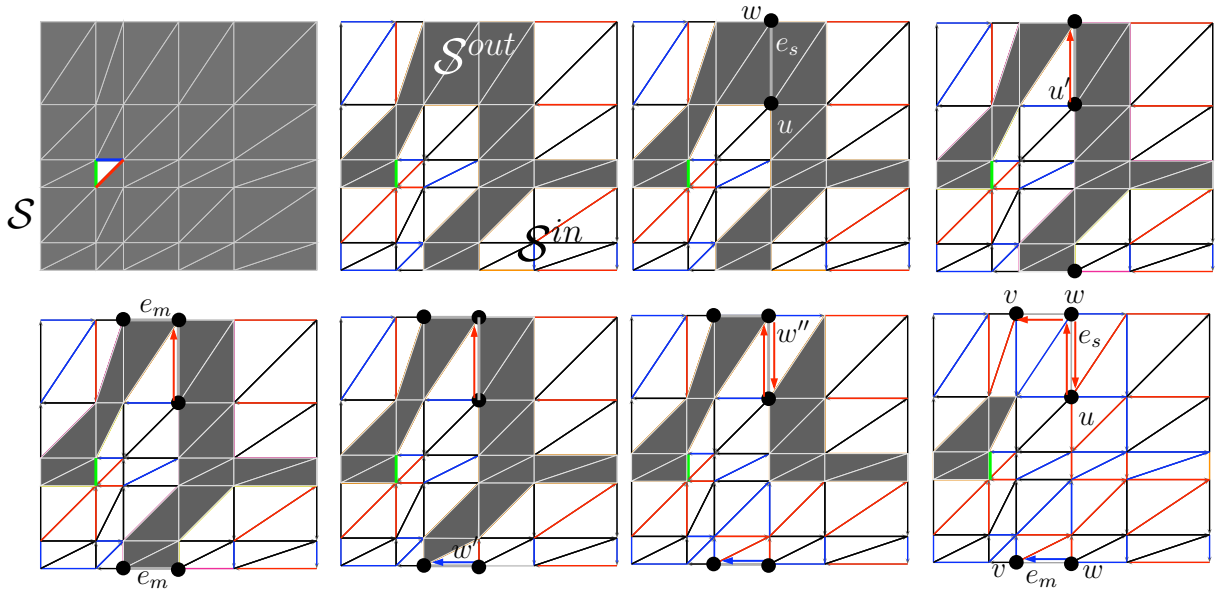
Figure 6 shows the traversal algorithm executed on a toroidal triangulation.

Notice the subtlety that, for nonzero genus, the multiple vertices, i.e., the vertices incident to merge/split edges, are conquered several times, as illustrated in Figure 7. Precisely, for a vertex  $v$  incident to  $k \geq 0$  merge/split edges, `conquer`( $v$ ) occurs  $k + 1$  times.

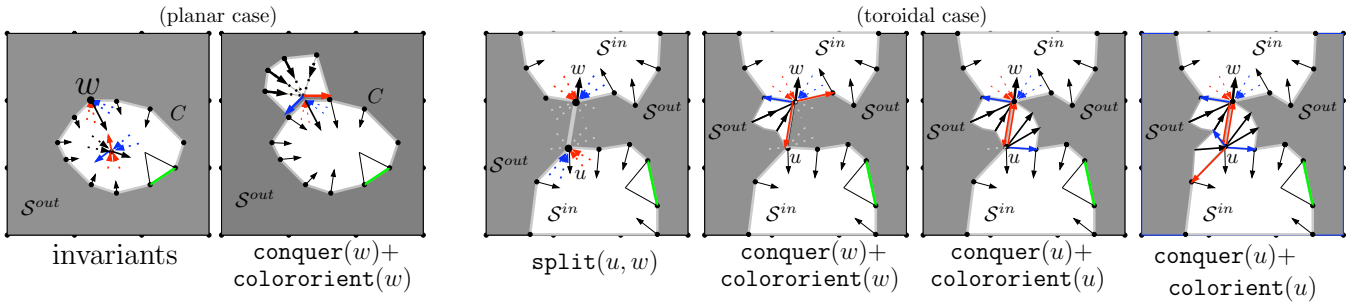
**Lemma 7** *For any triangulated surface  $S$  of genus  $g$ , COMPUTESCHNYDERANYGENUS( $S$ ) terminates, and can be implemented to run in linear time.*

*Proof:* (Elements of the proof) To prove that the traversal terminates, it suffices to show that a merge or split operation is always possible whenever there remain no free vertices on  $\partial S^{in}$ . To prove this, we use the property that a contractible chord for  $\partial S^{in}$  yields a free vertex on  $\partial S^{in}$  (precisely, a free vertex in the subpath  $P$  of  $\partial S^{in}$  such that  $P + e$  is a contractible cycle). Hence, if there remain no free vertices, then each vertex on  $\partial S^{in}$  is incident to a chordal edge, and all chordal edges are noncontractible. In that case, let  $v$  be a vertex on  $\partial S^{in}$  and let  $e$  be a chordal edge incident to  $v$ . If  $e$  is nonseparating, then we are fine. Otherwise, cutting  $S^{out}$  along  $e$  splits  $S^{out}$  into two smaller surfaces  $S_1$  and  $S_2$ . By an inductive argument, there exists a nonseparating chordal edge for  $\partial S_1$ , so we are done.

Implementing the algorithm to run in linear time requires some bookkeeping. For this purpose, we maintain the list of chordal edges for  $\partial S^{in}$ , and, which is more difficult, we



**Figure 6:** An example of execution of our traversal algorithm for a triangulated surface of genus 1. As in the planar case the traversal starts from the root face (the one incident to the green root edge). As far as only conquer operations on free vertices are performed, the area already explored (white triangles) remains planar and homeomorphic to a disk. When there remain no free vertices to conquer (second Figure), it is possible to find split and merge edges (incident to black circles). After performing split and merge operations, (here on the edges  $(u, w)$  and  $(w, v)$ ), new free vertices can be found (here  $u'$ ,  $w'$ , and  $w''$ ) in order to continue the traversal, until the entire graph is visited.



**Figure 7:** These pictures show the result of a *colorient* operation, both in the planar case (left picture) and higher genus case. Any chordal split or merge edge can be oriented in one or two directions (having possible two colours), depending on the traversal order on its extremities.

maintain a classification of these chordal edges as contractible, noncontractible separating, split edges, and merge edge. Such a classification is done by maintaining a spanning cut-graph  $G^c$  for the dual of  $S^{out}$ , such that  $G^c$  contains the dual of all chordal edges; and by maintaining a depth-first-search spanning tree of  $G^c$  so as to test which edges of  $G^c$  disconnect  $G^c$ . All this bookkeeping can be done in amortized linear cost. ■

**Theorem 8** *Let  $S$  be a triangulated surface of genus  $g$ . Then the coloration and orientation of edges obtained at the end of the traversal procedure  $\text{COMPUTESCHNYDERANYGENUS}(S)$  is a genus  $g$  Schnyder wood on  $S$ . The Schnyder wood thus obtained has the additional property that the graph  $T_2$  formed by the edges with color 2 is a tree, and the embedded graph formed by  $T_2$  and the  $2g$  special edges is a one-face map;*

*moreover the embedded graphs  $T_0, T_1$  formed respectively by the edges of color 0 and of color 1 are genus  $g$  maps with at most  $1 + 2g$  faces.*

*Proof:* (Elements of the proof) To prove that the combinatorial structure obtained by the traversal algorithm is a genus  $g$  Schnyder wood, it suffices to define some invariants that remain satisfied all along the traversal. We make use of the invariants already defined for the planar case, as well as new invariants relative to multiple vertices, which state that the local condition for multiple vertices is satisfied for those in  $S^{in} \setminus \partial S^{in}$ , and is already partially satisfied for those on  $\partial S^{in}$ .

Let us now sketch the proof of the second part of the theorem, i.e., the additional properties of  $T_0$ ,  $T_1$ , and  $T_2$ . Observe that, by design of the traversal algorithm,  $T_2$  is a connected (embedded) graph spanning all vertices of

$\mathcal{S} \setminus \{v_0, v_1\}$ . Moreover, all these vertices have outdegree 1 in  $T_2$ , except the root vertex  $v_{n-1}$  that has outdegree 0. Hence,  $T_2$  is a tree spanning the vertices of  $\mathcal{S} \setminus \{v_0, v_1\}$  and rooted at  $v_{n-1}$ . Moreover, the addition of each of the  $2g$  special edges creates a new fundamental cycle of the surface, so the embedded graph formed by  $T_2$  and the  $2g$  special edges is a one-face map. Concerning the edges of color  $i \in \{0, 1\}$ , an argument based on some topological invariants ensures that all the faces of the sets  $T_i$  are topological disks (hence defining a map). More precisely, by construction, a `conquer` operation does not change the topology of  $T_i$ , while a `split` or `merge` operation correspond to identify nodes in the three, which creates (by definition) new non-trivial cycles. Finally, the bound  $1 + 2g$  on the number of faces is due to the fact that each special edge contributes at most by one to the number of new faces of  $T_i$ . ■

**Remark 1** *The properties of  $T_0, T_1, T_2$  stated in Theorem 8 can be considered as extensions of the fundamental property of planar Schnyder woods, which states that  $T_0, T_1$ , and  $T_2$  are plane trees. Figure 8 shows an example in genus 1.*

#### 4 Application to encoding triangulated surfaces of genus $g$ , via one-face maps

In the planar case, Schnyder woods yield a simple encoding procedure for triangulations. From the observation that the trees  $T_1$  and  $T_2$  of a Schnyder wood are enough to recover the whole Schnyder wood (hence, the underlying triangulation as well), the coding word is just made of the pair  $(W_1, W_2)$  of parenthesis words encoding respectively the trees  $T_1$  and  $T_2$ . Hence, for a triangulation with  $n + 3$  vertices, the coding word has length  $4n$ . This code is both simple and quite compact, as the length  $4n$  is not far from the information-theory lower bound of  $\log_2(4^4/3^3)n$  ( $\log_2(4^4/3^3) \approx 3.245$ ), which is attained in the planar case by a bijective construction due to Poulalhon and Schaeffer [27].

Such a bijective construction is still to be found for higher genus, but we can here at least extend to higher genus the simple encoding procedure based on Schnyder woods. First, it is easily checked that a genus  $g$  Schnyder wood is recovered from the knowledge of the map  $T_1$ , the plane tree  $T_2$ , and the location of the special edges and vertices in  $T_1$  and  $T_2$ . As  $T_1$  has at most  $1 + 2g$  faces, it is encoded as the parenthesis word of a plane spanning tree +  $O(g)$  pointers to encode the edges of  $T_1$  not in the spanning tree. All in all, we obtain the following result:

**Proposition 9** *A triangulated surface of genus  $g$  with  $n + 3$  vertices can be encoded in linear time —via a genus  $g$  Schnyder wood— by two parenthesis words each of length  $2n$ , plus  $O(g)$  pointers each of length  $O(\log(n))$ ; so the total code length is  $4n + O(g \log(n))$ . Coding and decoding can be done in linear time.*

We can also design a more sophisticated code that supports queries: this time we need to encode the three maps  $T_i$  using multiple parenthesis systems (as done in [11, 1]).

**Proposition 10** *Given  $\mathcal{S}$  a triangulated surface of genus  $g$  with  $e$  edges, there exists a compact representation of  $\mathcal{S}$  using asymptotically  $(2 \log_2 6) \cdot e + O(g)$  bits which answers queries for vertex adjacency and vertex degree in  $O(1)$  time. Coding and decoding take linear time.*

#### 5 Conclusion and perspectives

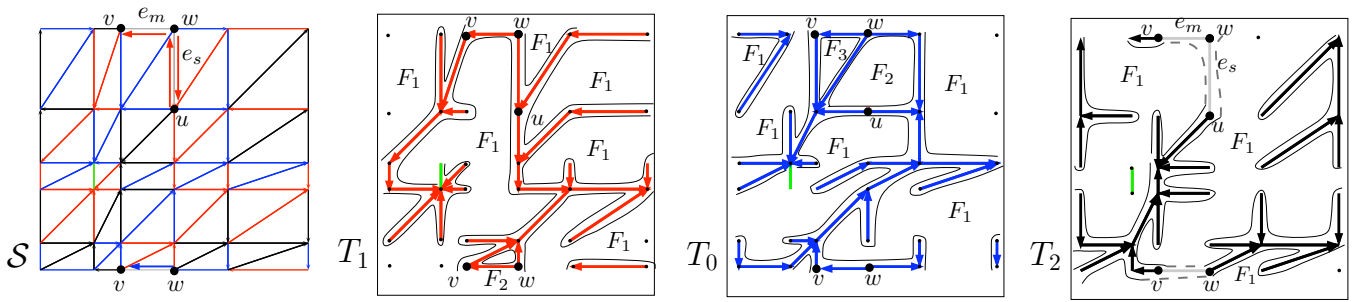
We have presented a general approach for extending to higher genus a fundamental combinatorial structure, Schnyder woods, which is by now a standard tool to handle planar graphs both structurally and algorithmically. We have been successful in showing that the definition and several fundamental combinatorial properties can be extended from the planar to the genus  $g$  case in a natural way. We point out that our graph traversal procedure induces an ordering for treating the vertices so as to shell the surface progressively. Such an ordering is already well known in the planar case under the name of *canonical ordering* and has numerous applications for graph encoding and graph drawing [11, 19]. It is thus of interest to extend this concept to higher genus. The only difference is that in the genus  $g$  case there is a small number —at most  $2 \cdot 2g$ — of vertices that might appear several times in the ordering; these correspond to the vertices incident to the  $2g$  special edges (split/merge edges) obtained during the traversal.

**Further extensions.** Our approach relies on quite general topological and combinatorial arguments, so the natural next step should be to apply our methodology to other interesting classes of graphs (not strictly triangulated), which have similar characterization in the planar case. Our topological traversal could be extended to the 3-connected case, corresponding to polygonal meshes of genus  $g$ . We point out that our encoding proposed in Section 4 could take advantage of the existing compact encodings of planar graphs [11, 10, 18], using similar parenthesis-based approaches.

**Lattice structure.** There are a number of deep and interesting questions which our work leaves open. From the combinatorial point of view it should be of interest to investigate whether edge orientations and colourations in genus  $g$  have nice lattice properties, as in the planar case. In particular, the existence of minimal orientations in the planar case led to a large number of deep results in areas such as graph drawing, optimal sampling and coding.

**Graph encoding applications** In addition to the new tree decomposition proposed for the higher genus case, one contribution has been to show, from the algorithmic point of view, that vertex-decomposition based traversal can be performed in linear time for any genus  $g \geq 0$ . In particular, the linear time complexity of our traversal procedure is due to the very low complexity,  $O(1)$  time, to check that a chordal edge is noncontractible. Thus a “left-most” traversal of the triangulated surface should be doable in linear time (let us recall that a left-most driven traversal returns in the planar case the minimal orientations mentioned above). This remark, together with the local properties of genus  $g$  Schnyder woods





**Figure 8:** A triangulated torus endowed with a Schnyder wood. The black edges form a tree  $T_2$ , and the addition of the two special edges yields a one-face map; The embedded graph  $T_0$  (blue edges) is a map with 3 faces; and the embedded graph  $T_1$  (red edges) is a map with 2 faces.

(Definition 2), suggest the possibility of further nice applications in graph encoding and sampling (see [16, 27] for the planar case). More precisely, we conjecture the existence of correspondences between some special classes of one-face maps (generalizations of plane trees) having a constant fixed number of leaves per node (almost everywhere), and the popular classes of triangulations and 3-connected maps of genus  $g$ : this would lead to the first (linear time) optimal compression schemes and succinct encodings for these class of objects.

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