Combining points and tangents into parabolic polygons: an affine invariant model for plane curves

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Abstract. Image and geometry processing applications estimate the local geometry of objects using information localized at points. They usually consider information about the tangents as a side product of the points coordinates. This work proposes parabolic polygons as a model for discrete curves, which intrinsically combines points and tangents. This model is naturally affine invariant, which makes it particularly adapted to computer vision applications. As a direct application of this affine invariance, this paper introduces an affine curvature estimator that has a great potential to improve computer vision tasks such as matching and registering. As a proof–of–concept, this work also proposes an affine invariant curve reconstruction from point and tangent data.
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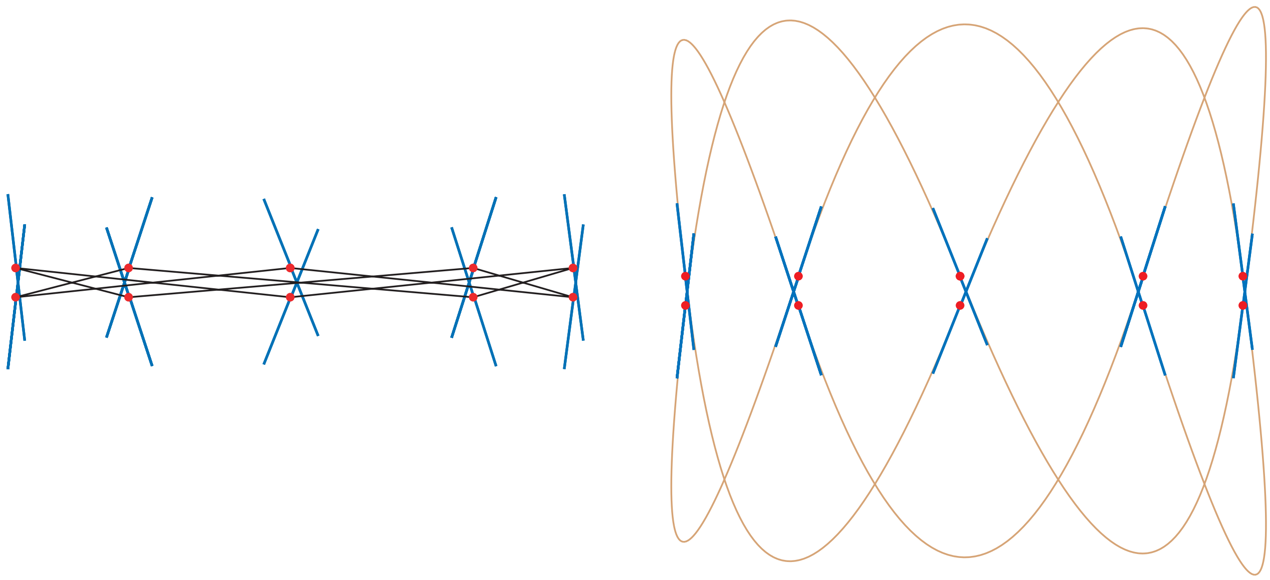

Figure 1: *Parabolic polygon (right) obtained from a Lissajous curve with 10 samples vs. straight line polygon ignoring tangents (left).*

1 Introduction

Computers represent geometric objects through discrete structures. These structures usually rely on point-wise information combined with adjacency relations. In particular, most geometry processing applications require the normal of the object at each point: either for rendering [15], deformation [4], or numerical stability of reconstruction [1]. Modern geometry acquisition processes for curves or surfaces usually provide measures of the normals together with the point measures. These normals can also be robustly estimated only from the point coordinates [11, 9], or from direct image processing [5, 16].

However, the normal or tangent information is usually considered separately from the point coordinate, and the definition of geometrical objects such as contour curves or discrete surfaces depends rather on the point coordinates. Although modeling already makes intensive use of this information, in particular with Bézier curves, only recent developments in reconstruction problems proposed to incorporate these tangents as part of the point set definition [1].

This work proposes a discrete curve representation that intrinsically combines points and tangents: the parabolic polygons (Figure 1), introduced in section 3 *Parabolic poly-*

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gons. This model is naturally invariant with respect to affine transformations of the plane. This makes it particularly adapted to computer vision applications, since two contours of the same planar object obtained from different perspectives are approximately affine equivalent [17]. For example, based on this parabolic polygon, estimators for the affine length and the affine curvature are defined (see section 4 *Affine Estimators*). These estimators, being affine invariant, have a great potential to improve computer vision tasks like matching and registration [12].

The affine length estimator considered in this paper is well-known: It has been adapted from [13, 14], and shown to be optimal in [10]. With respect to affine curvature estimators, the only works we are aware dealing with this issue are [13, 14] and [2]. They all estimate affine curvature from five consecutive samples only in convex position, interpolating them by a conic. The affine curvature estimator that is proposed in this paper is more concise. It estimates the affine curvature from just three consecutive samples, which is well suited for applications such as reconstruction, interpolation and blending. The convergence properties of both estimators are described at section 4 *Affine Estimators*.

The application to curve reconstruction presented in section 5 *Affine Curve Reconstruction* is a variation of [7], where the Euclidean distance computation is replaced by affine estimates. This leads to an affine invariant curve reconstruction, which works well at least on synthetic examples. Moreover, it was observed that the introduction of the curvature in the algorithm significantly improves the stability of the reconstruction, thus pledging for the validity of the affine curvature estimator.

This paper is an extension of [6], where we handled mainly convex curves. The main new aspect of the present paper is the inclusion of double parabolic connections in the construction of parabolic polygons for non-convex curves, and the corresponding extensions of the reconstruction algorithm. Subsequently a local, affine invariant criterion for locating these double parabolic connections is introduced. This new aspect, although delicate in affine geometry, is critical to the practical use of the proposed model in computer vision.

2 Review of Affine Geometry

This section quickly recalls the definitions of affine quantities that are relevant to this work. The reader can find a detailed presentation of affine geometry of plane curves in Buchin's book [3].

Affine invariance. Consider a smooth curve γ in the plane and A an arbitrary linear transformation of the plane with determinant 1. A scalar function g on γ is *affine invariant* if, for every $p \in \gamma$, $g(A(p)) = g(p)$. Similarly, a vector-valued V on γ is *affine invariant* if, for every $p \in \gamma$, $V(A(p)) = A \cdot V(p)$. This notion is more precisely referred to as *equiaffine* invariant, ant it is the only affine invariance considered in this paper. If one considers also invertible linear transformations with arbitrary determinant, then the transformed functions are multiplied by constants.

(a) Basic affine invariant quantities

Affine length. Consider a curve γ parameterized by $\mathbf{x}(t)$, $t_0 \le t \le t_1$, and assume that it is convex, i.e., that $\mathbf{x}'(t) \wedge \mathbf{x}''(t)$ does not change sign, where the cross product $X \wedge Y$ denotes the determinant of the 2×2 matrix whose columns are the vectors X and Y. Assuming that $\mathbf{x}'(t) \wedge \mathbf{x}''(t) > 0$, the number

$$s(t) = \int_{t_0}^{t} \left(\mathbf{x}'(t) \wedge \mathbf{x}''(t)\right)^{\frac{1}{3}} dt$$

is called the affine parameter and $L = s(t_1) - s(t_0)$ is called the affine length of the curve.

Affine tangent and normal. The affine invariant vectors $\mathbf{v}(s) = \mathbf{x}'(s)$ and $\mathbf{n}(s) = \mathbf{x}''(s)$ are called *affine tangent* and *affine normal*, respectively. The affine tangent is tangent to the curve, but the affine normal is not necessarily perpendicular to the curve in the Euclidean sense. In fact, these vectors are characterized by the equation

$$\mathbf{v}(s) \wedge \mathbf{n}(s) = 1. \qquad (1)$$

Affine curvature. Differentiating equation (1), one can see that $\mathbf{x}'(s)$ and $\mathbf{x}'''(s)$ are co-linear : $\mathbf{x}'''(s) = -\mu(s)\,\mathbf{x}'(s)$. The factor $\mu(s)$ is called the *affine curvature*. Equivalently, one can define the affine curvature by $\mu(s) = \mathbf{x}''(s) \wedge \mathbf{x}'''(s)$.

Comparison with Euclidean geometry. In Euclidean geometry, points have zero length, lines have zero curvature and circles have constant curvature. By comparison, in affine geometry, lines have zero length, parabolas have zero curvature and conics have constant curvature.

(b) Affine behavior close to inflection points

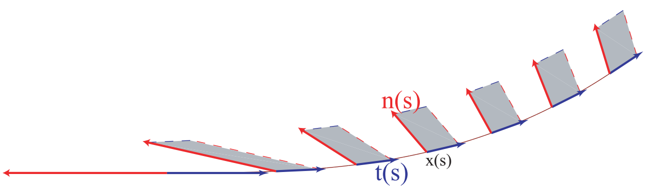


Figure 2: *Affine tangents and normals close to an inflection point.*

In order to extend [6] to non–convex curves, it is important to understand the behavior of the affine quantities near a higher order tangent. Considering a convex arc $\mathbf{x}(s)$ beginning at an inflection point $\mathbf{x}(0)$, its affine length remains finite close to the inflection, while the affine curvature $\mu(s)$ tends to ∞, when $s \to 0$, and $\int_0^{s_1} \mu(s)\, ds = \infty$ for any $s_1 > 0$.

This behavior can be well observed on the curve $\mathbf{x}(t) = (t, t^n)$, $n \ge 3$, $0 \le t \le 1$, which has a higher order tangent at $t = 0$. Easy calculations lead to an affine parameterization of the curve:

$$\mathbf{x}(s) = \left(cs^{\frac{3}{n+1}}, c^n s^{\frac{3n}{n+1}}\right) \quad \text{with} \quad c = \left(\frac{(n+1)^3}{27n(n-1)}\right)^{\frac{1}{n+1}}.$$

The affine tangent and normal are obtained by derivation:

$$\mathbf{v}(s) = \left(\frac{3c}{n+1}\, s^{\frac{2-n}{n+1}}, \frac{3nc^n}{n+1}\, s^{\frac{2n-1}{n+1}}\right) \quad \text{and}$$

$$\mathbf{n}(s) = \left(\frac{3c\,(2-n)}{(n+1)^2}\, s^{\frac{1-2n}{n+1}}, \frac{3nc^n\,(2n-1)}{(n+1)^2}\, s^{\frac{n-2}{n+1}}\right).$$

Observe that the affine tangent tends to an infinite length vector in the positive x-direction, when $s \to 0$, while the affine normal tends to an infinite length vector in the x-direction, but in the negative sense (see Figure 2). The affine curvature is given by

$$\mu(s) = \frac{(n-2)(2n-1)}{(n+1)^2}\, s^{-2}.$$

Hence $\mu(s) \to \infty$, when $s \to 0$, and $\int_0^{s_1} \mu(s)\, ds = \infty$, for any $s_1 > 0$.

3 Parabolic polygons

A discrete curve model usually consists of a discrete set of ordered samples in the plane. When each sample carries only its coordinates, straight line polygons are natural continuous representations for it, since line segments have Euclidean curvature 0. When the samples carry also the information of the direction of the tangent line, parabolas are a natural connection between two consecutive samples since they have null affine curvature.

To be more precise, consider ordered samples $(\mathbf{x}_i, l_i)$, $1 \leq i \leq n$, where l_i is a line passing through point $\mathbf{x}_i$ representing the tangent (see Figure 1(left)). In this section we shall define the *parabolic polygon* of these samples, which is a curve consisting of arcs of parabolas passing through the points and tangent to the lines of the samples (see Figure 1(right)).

The parabolic polygon is constructed in two steps (Figure 3): The basic parabolic polygon (section 3(a) *Basic parabolic polygon*) is a concatenation of single parabolas between consecutive samples. However, this concatenation may generate incoherencies in the tangent orientation. These incoherencies can be corrected by choosing one of the parabolas and replacing it by a double parabolic connection (section 3(b) *Double parabolic connections*).

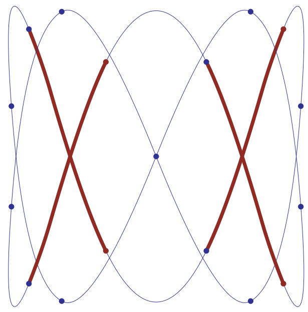


Figure 3: *A parabolic polygon: the basic parabolas in blue, with the double parabolic connection in brown. Observe that the basic part contains inflection points.*

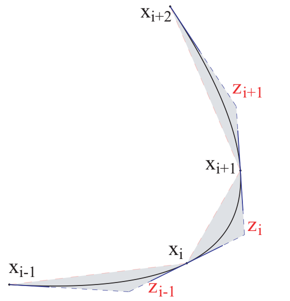


Figure 4: *The support points and triangles of a parabola sampling.*

(a) Basic parabolic polygon

Parabola between two samples. Let us first construct a continuous interpolation between two consecutive samples $(\mathbf{x}_i, l_i)$ and $(\mathbf{x}_{i+1}, l_{i+1})$. If l_i is not parallel to l_{i+1}, there exists a unique parabola P_i passing through $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ and tangent to l_i and l_{i+1} at these points. Denote by $\mathbf{z}_i$ the point of intersection of the lines l_i and l_{i+1}. The triangle whose vertices are $\mathbf{x}_i$, $\mathbf{z}_i$ and $\mathbf{x}_{i+1}$ is called the *support triangle* (see Figure 4). The orientation of the support triangle is defined by the sign of $\Delta_i = (\mathbf{z}_i - \mathbf{x}_i) \wedge (\mathbf{x}_{i+1} - \mathbf{z}_i)$, and its area will be denoted by A_i.

If the orientation of the support triangle is positive, the affine tangent vectors of P_i at points $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ are respectively

$$\mathbf{v}_{i,1} = -\frac{2}{L_i}(\mathbf{x}_i - \mathbf{z}_i) \quad \text{and} \quad \mathbf{v}_{i,2} = \frac{2}{L_i}(\mathbf{x}_{i+1} - \mathbf{z}_i)$$

where $L_i = 2A_i^{1/3}$ is the affine length of P_i. If the orientation of the support triangle is negative, the signs must be interchanged. In both cases, the affine normal and the affine parameterization of P_i are given by

$$\mathbf{n}_i = \frac{2}{L_i^2}(\mathbf{x}_i + \mathbf{x}_{i+1} - 2\mathbf{z}_i) \quad \text{and} \quad \gamma_i(s) = \mathbf{x}_i + s\mathbf{v}_{i,1} + \frac{s^2}{2}\mathbf{n}_i.$$

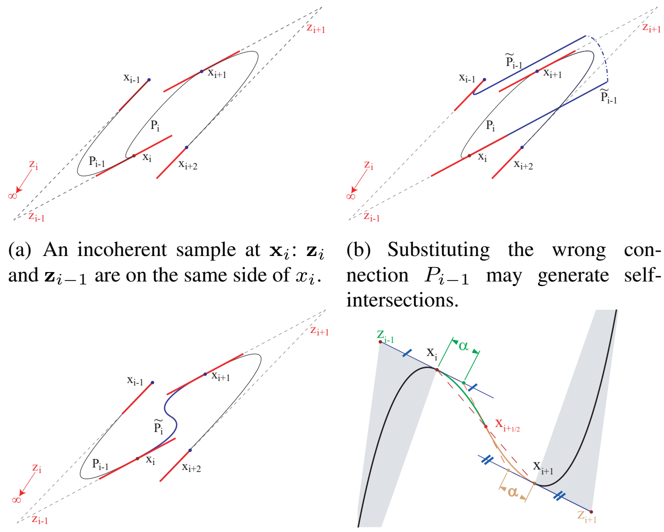
Coherence in parabolas concatenation. To concatenate properly two parabolas P_{i-1} and P_i, the tangent vectors at the common point $\mathbf{x}_i$ must have the same orientation. This coherence can be checked by the relative positions of points $\mathbf{z}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{z}_i$ in the line l_i (see Figure 4). Denote by λ_i be the collinearity factor,

$$\mathbf{z}_i - \mathbf{x}_i = \lambda_i(\mathbf{x}_i - \mathbf{z}_{i-1}).$$

If $\lambda_i > 0$, sample i will be called *coherent*. If $\lambda_i < 0$, then sample i will be called *incoherent* (see Figure 5(a)). Observe that, if the curve is convex, all the samples are coherent, although the concatenation of parabolas at coherent samples may also contain inflection point, as shown on Figure 3.

(b) Double parabolic connections

Simplifying hypothesis. This section describes how to handle incoherent parabolas or parallel tangent lines at consecutive samples. In what follows, we shall assume that these problems do not occur for three consecutive samples. This hypothesis is not essential, but it makes the description of the model close to inflection points easier. Besides, if the samples are obtained with sufficiently density from a smooth curve with a finite number of inflection points, the hypothesis is valid.



(a) An incoherent sample at $\mathbf{x}_i$: $\mathbf{z}_i$ and $\mathbf{z}_{i-1}$ are on the same side of x_i.
(b) Substituting the wrong connection P_{i-1} may generate self-intersections.
(c) Substituting the right connection P_i.
(d) Double parabola construction.

Figure 5: *Criterion to construct connections from an incoherent sample: substituting the wrong connection by a double parabolic arc may create self-intersection. A pair of parabolas at the inflection point can then be constructed in an affine invariant manner.*

Connections at incoherent samples. Incoherent samples always appear close to an inflection point of the original curve. If $\lambda_i < 0$, the parabolas P_{i-1} and P_i cannot be concatenated. One should then substitute one of the parabolas, P_{i-1} or P_i, by a pair of parabolas concatenated at a virtual inflection point that will be called a *double parabolic connection*.

The criterion to decide which parabola to discard is illustrated on Figure 5: If $\mathbf{z}_i$ and $\mathbf{x}_{i-1}$ are on different sides of the line connecting $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$, then one substitutes P_{i-1}. If they are in the same side, then one substitutes P_i. This procedure was proposed in order to avoid local self-intersections of the curve.

Constructing double parabolic connections. To construct the double parabolic connection between P_{i-1} and P_{i+1}, one must find a reasonable point-tangent position to represent an inflection point and its tangent. The choice of the exact inflection point has too many degrees of freedom to be determined only from $(\mathbf{x}_i, l_i)$ and $(\mathbf{x}_{i+1}, l_{i+1})$. We thus propose here a simple, affine invariant heuristic based on the previous and next sample (see Figure 5(d)).

Consider the support points $\mathbf{z}_{i-1}$ and $\mathbf{z}_{i+1}$ and define $\overline{\mathbf{z}}_{i-1} = \mathbf{x}_i + \alpha(\mathbf{x}_i - \mathbf{z}_{i-1})$ and $\overline{\mathbf{z}}_{i+1} = \mathbf{x}_{i+1} + \alpha(\mathbf{x}_{i+1} - \mathbf{z}_{i+1})$, where α is fixed to 0.46, which approximates optimally a regularly sampled cubic (see Figure 5(d)). The point of intersection of the diagonals of the quadrilateral $\mathbf{x}_i \overline{\mathbf{z}}_{i-1} \mathbf{x}_{i+1} \overline{\mathbf{z}}_{i+1}$ is denoted $\mathbf{x}_{i+\frac{1}{2}}$ and the line that passes through $\overline{\mathbf{z}}_{i-1}$ and $\overline{\mathbf{z}}_{i+1}$ is denoted $l_{i+\frac{1}{2}}$. The double parabolic polygon is then the concatenation of the parabola defined by $(\mathbf{x}_i, l_i)$ and $(\mathbf{x}_{i+\frac{1}{2}}, l_{i+\frac{1}{2}})$ with the parabola defined by $(\mathbf{x}_{i+\frac{1}{2}}, l_{i+\frac{1}{2}})$ and $(\mathbf{x}_{i+1}, l_{i+1})$.

We can summarize the double parabolic connection procedure as follows:

1. *[Coherence test]* Compute the sign of λ_i for each sample $(\mathbf{x}_i, l_i)$;
2. *[Valid parabola test]* When $\lambda_i < 0$, if $(\mathbf{z}_{i-1} - \mathbf{x}_i) \wedge (\mathbf{x}_i - \mathbf{x}_{i-1})$ and $(\mathbf{x}_{i+1} - \mathbf{x}_i) \wedge (\mathbf{x}_i - \mathbf{x}_{i-1})$ have opposite signs, discard P_{i-1}. Otherwise discard P_i.
3. *[Double parabola construction]* For each discarded parabola P_i, insert the sample $(\mathbf{x}_{i+\frac{1}{2}}, l_{i+\frac{1}{2}})$ between samples i and $i+1$.

4 Affine Estimators

This section proposes an affine length estimator for any smooth curve and an affine curvature estimator for convex curves, both based on the parabolic polygon approximation of the curve. We further study the convergence of these estimators from the theoretical and experimental points of view.

(a) The affine length and affine curvature estimators.

The affine length of a parabolic polygon P is the sum of the affine lengths L_i of the parabolic arcs P_i, where L_i was defined from the area of the support triangle at section 3(a) *Basic parabolic polygon*. This affine length can be used as an estimate of the affine length of any curve. The affine invariance of this affine length estimator can be seen in Figure 6.

To obtain an estimator for the integral of the affine curvature of a convex curve C, observe that the affine curvature integral can be approximated by

$$\begin{aligned}
\mu(P) &= \int_C \mu ds = \int_C (\mathbf{n}'(s) \wedge \mathbf{n}(s))\, ds \\
&\approx \sum \left(\frac{\mathbf{n}(s+\Delta s) - \mathbf{n}(s)}{\Delta s} \wedge \mathbf{n}(s) \right) \Delta s \\
&\approx \sum \mathbf{n}(s+\Delta s) \wedge \mathbf{n}(s) = \sum_{i=2}^{n-1} \mathbf{n}_{i-1} \wedge \mathbf{n}_i .
\end{aligned}$$

For a parabolic polygon, the affine curvature is zero except at the samples, where it is concentrated. One can define the

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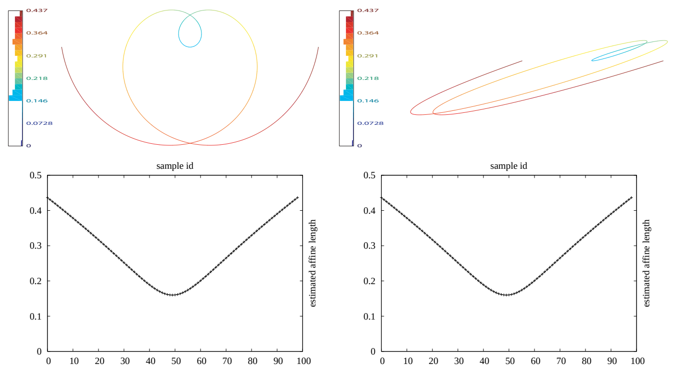


Figure 6: *Affine length estimations on a spiral with 100 samples, before and after an affine transformation.*

affine curvature at a convex sample i by

$$\mu_i = \frac{\mathbf{n}_{i-1} \wedge \mathbf{n}_i}{\frac{1}{2}\left(L_{i-1} + L_i\right)}.$$

The affine curvature at an inflection point is ∞. These estimators have the important property of being affine invariant (see Figure 7).

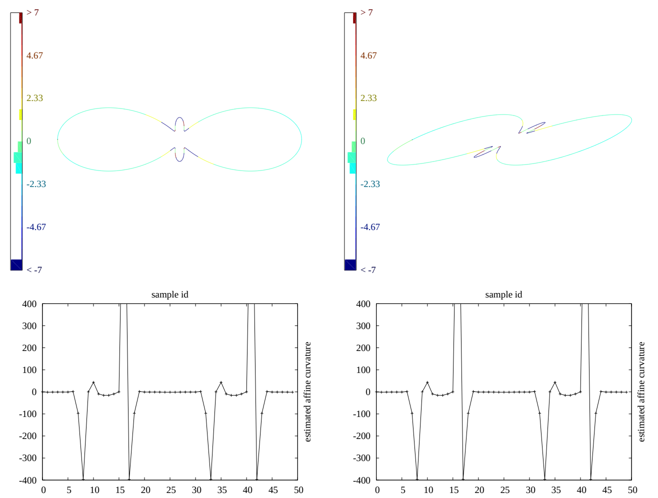


Figure 7: *Affine estimations on a lemniscate with 51 samples, before and after an affine transformation.*

(b) Convergence: Theoretical results

Consider a convex arc C in the plane. Let $S = \big((\mathbf{x}_i, l_i)\big), 1 \le i \le n$ be a sampling of C, where l_i is the line tangent to C at $\mathbf{x}_i$. Denote by h the maximum affine length between sample points along C. We say that the affine length estimator is convergent if $L(S) = \sum_{i=1}^{n-1} L_i$ converges to the affine length L of the curve, when $h \to 0$. Similarly, the affine curvature estimator is said to be convergent if $\mu(S) = \sum_{i=2}^{n-1} \mathbf{n}_{i-1} \wedge \mathbf{n}_i$ converges to $\int_C \mu(s)\, ds$, when $h \to 0$.

In [13], it is shown the convergence of the affine length estimator. The same paper proves that if $S_1 \subset S_2$, then $L(S_1) > L(S_2)$. Moreover, [10] proves that this estimator is of order 4: $|L(S) - L| = O(h^4)$. In the rest of this subsection we prove the convergence of the affine curvature in the particular case of constant affine curvature curves and equally spaced samples. A general proof in the convex case can be found at `http://www.mat.puc-rio.br/~tomlew/convergence_puc.pdf`.

Constant positive curvature. In this example, we consider the case of a curve with constant positive affine curvature. By making an affine transformation of the plane, we can assume that this curve is a circle. Consider points (x_i, y_i), $1 \le i \le n$, in a circle of radius R at an affine distance $s = L/n$, where $L = 2\pi R^{\frac{2}{3}}$ is the affine length of the circle [3]. The affine curvature of this circle is $\mu = R^{-\frac{4}{3}}$. The central angle determined by two consecutive points is $2\alpha = \frac{2\pi}{n}$.

Simple calculations shows that the affine length of the arc of parabola P_i is given by

$$L_i = \frac{2R^{\frac{2}{3}} \sin\alpha}{(\cos\alpha)^{\frac{1}{3}}}$$

and that the affine normal is orthogonal to the chord connecting (x_i, y_i) and (x_{i+1}, y_{i+1}), with norm $\|\mathbf{n}_i\| = (R\cos\alpha)^{-\frac{1}{3}}$. Thus the estimated affine curvature is given by $\mathbf{n}_i \wedge \mathbf{n}_{i+1} = (R\cos(\alpha))^{-\frac{1}{3}} \sin(2\alpha)$. The estimated affine length of the circle is then

$$\sum_{i=1}^{n-1} L_i = 2R^{\frac{2}{3}}(n-1)\frac{\sin\left(\frac{\pi}{n}\right)}{\left(\cos\left(\frac{\pi}{n}\right)\right)^{\frac{1}{3}}}$$

which converges to the affine length of the circle when $n \to \infty$. And the estimated affine length

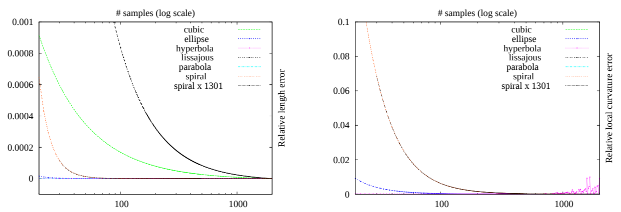
$$\mu(P) = (n-2)\left(R\cos\left(\frac{\pi}{n}\right)\right)^{-\frac{2}{3}} \sin\left(\frac{2\pi}{n}\right)$$

converges to $2\pi R^{-\frac{2}{3}} = L\mu$, when $n \to \infty$.

Constant negative curvature. In this example, we consider the case of a curve with constant negative affine curvature. By making an affine transformation of the plane, we can assume that this curve is a hyperbola $xy = c$, for some $c > 0$ [3]. Consider points (x_i, y_i), $1 \le i \le n$, in the hyperbola at an affine distance $s = L/n$, where $L = (2c)^{\frac{1}{3}} \ln(x_n/x_1)$ is the affine length of the arc of hyperbola between (x_1, y_1) and (x_n, y_n). The affine curvature of this hyperbola is $\mu = -(2c)^{-\frac{2}{3}}$.

Denote by $r = \frac{x_{i+1}}{x_i} = \frac{y_i}{y_{i+1}}$. From the fact that the affine lengths between (x_i, y_i) and (x_{i+1}, y_{i+1}) along the hyperbola is $(2c)^{\frac{1}{3}} \ln(r)$, one conclude that r does not depend on i. Straightforward calculations shows that the area of the support triangle defined by (x_i, y_i) and (x_{i+1}, y_{i+1}) is given by $c\,\frac{(r-1)^3}{2r(r+1)}$ and so the affine length of P_i is given by

$$L_i = \left(\frac{4c}{(r+1)r}\right)^{\frac{1}{3}}(r-1).$$



(a) Relative length error (b) Relative curvature error at mid-point

Figure 8: *Convergence of the affine estimators when the number of samples grows.*

Also, the affine normal to P_i is given by

$$\mathbf{n}_i = \left(\frac{r^2}{2(r+1)c^2}\right)^{\frac{1}{3}} (x_i, y_{i+1}),$$

thus

$$\mathbf{n}_{i-1} \wedge \mathbf{n}_i = \left(\frac{r+1}{4cr^2}\right)^{\frac{1}{3}} (1-r).$$

We conclude that

$$\sum_{i=1}^{n-1} L_i = (n-1)\left(\frac{4c}{(r+1)r}\right)^{\frac{1}{3}} (r-1)$$

converges to L. And that the estimated affine curvature of the arc

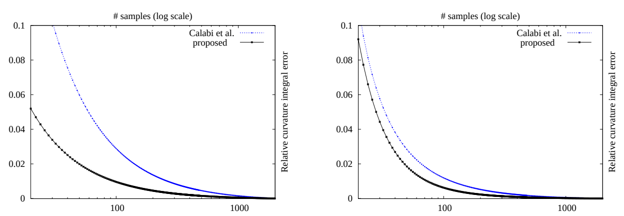
$$\sum_{i=2}^{n-1} \mathbf{n}_{i-1} \wedge \mathbf{n}_i = (n-2)\left(\frac{r+1}{4cr^2}\right)^{\frac{1}{3}} (1-r)$$

converges to $(2c)^{-\frac{1}{3}} \ln(x_n/x_1) = L\mu$.

(c) Convergence: Experimental results

The proposed estimators were tested with samples of smooth curves, sometimes with isolated singular points. The affine lengths were estimated for all curves, but the total curvature, $\int_C \mu ds$, only for convex curves. The local curvature at a fixed point was calculated for all curves. The results confirm the theoretically proved convergence of the affine length and curvature estimators (see Figure 8).

Calabi *et al.* [13] estimate, from 5 points, the affine curvature at the central point, while in our method, this estimation is obtained from 3 pairs $(\mathbf{x}_i, l_i)$. The experimental comparison of both methods is shown in Figure 9. In these experiments, our method had shown a faster convergence.



(a) Hyperbola (b) Spiral

Figure 9: *Comparison of the relative total curvature error using [13] and our technique.*

5 Affine Curve Reconstruction

This section proposes a curve reconstruction algorithm from points and tangents to test the validity of the proposed model and estimators on a simple and practical application. This is the typical case, for example, of a shape extracted by edge detection from an image, or from direct measures produced by laser scanner. Given a finite set of point-line samples $\{(\mathbf{x}_1, l_1), \cdots, (\mathbf{x}_n, l_n)\}$, where l_i represent the tangent line at point $\mathbf{x}_i$, how to sort these pairs to obtain a parabolic polygon representing a "reasonable" curve? If the samples are obtained from a smooth curve, one expects the reconstructed parabolic polygon to be "close" to the original curve. The solution proposed here uses intrinsically the tangent information and is affine invariant. It is a greedy method adapted from [7] for the parabolic polygon model. It further uses our affine curvature estimator in a way similar to [8].

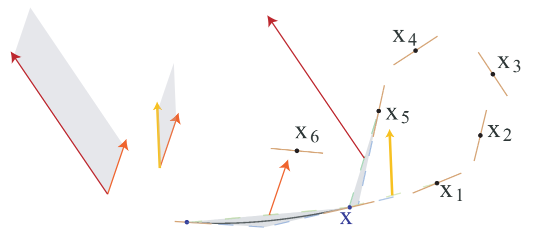


Figure 10: *Point $\mathbf{x}_5$ is at the smallest affine distance of $\mathbf{x}$, but it is rejected because it would induce a big affine curvature (left area), while connection with $\mathbf{x}_1$ induces a smaller curvature (right area).*

(a) The reconstruction algorithm

The algorithm starts with the pair $((\mathbf{x}_i, l_i), (\mathbf{x}_j, l_j))$ which have the smallest affine distance. The algorithm then proceeds greedily, looking for the pair $(\mathbf{x}_k, l_k)$ which is at a minimum affine distance of $(\mathbf{x}_j, l_j)$, with three restrictions:

1. it avoids samples which were already connected to two other samples ;
2. it rejects big changes in affine length, say $L(\mathbf{x}_j, \mathbf{x}_k) > rL(\mathbf{x}_i, \mathbf{x}_j)$, for some $r > 0$;
3. it rejects big affine curvatures, say $\mu(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k) > \kappa$, for some $\kappa > 0$.

This last condition is added to avoid undesirable deviations as illustrated on Figure 10, but it can also indicate proximity of an inflection point. Observe that the affine length computation uses the double parabolic connection in case of incoherence at a sample. The algorithm can get stuck when all the free samples would induce big curvatures or big affine length. In such case, the algorithm returns to the initialization phase, looking for the smallest available connection.

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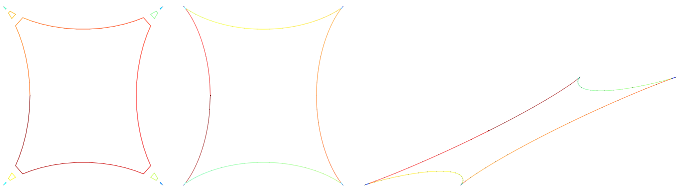
(b) Reconstruction results on synthetic examples

Figure 11 provides a visual comparison between the proposed algorithm and the Euclidean reconstruction algorithm. The algorithm used for the Euclidean reconstruction is based on [7]. As expected, the use of the tangent information improves dramatically the result. It is also interesting to observe the affine invariance of the proposed algorithm. The effect of the parameter k, that controls undesirable deviations, can be observed in figures Figure 12 and Figure 13.

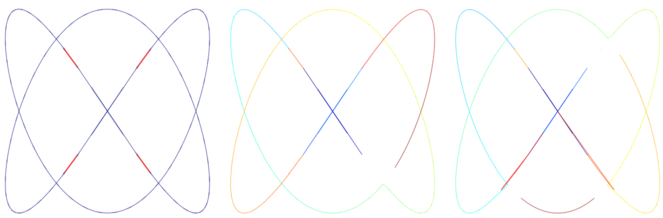


(a) Euclidean (b) affine (c) affine

Figure 11: *Euclidean and affine reconstructions of a singular curve, before and after an affine transformation: colors indicate the reconstruction order, from blue to red.*

6 Conclusion and Future Work

Conclusion. In this work the parabolic polygon is proposed as a model for discrete curves that takes into account the position and the tangent line at each sample. Based on this model, an affine length estimator and an affine curvature estimator are proposed. The model and the estimators are affine invariant, which shows they have a great contribution potential in computer vision tasks like matching and registering. This work proposes also a curve reconstruction algorithm based on the model. The validity of the affine length and affine curvature estimators were checked in this algorithm.

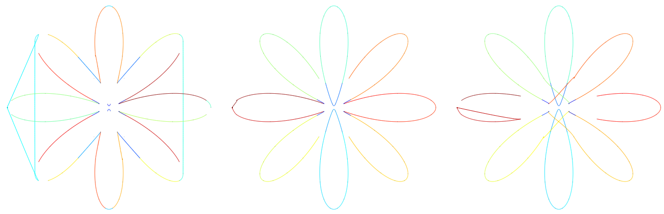


(a) $\kappa = 1$ (b) $\kappa = 10$ (c) $\kappa = 100$

Figure 12: *Influence of the curvature threshold on the reconstruction of a Lissajous curve.*

Future work. From the theoretical point of view, an interesting problem, related to the use of projective differential invariants in computer vision, is to look for the delicate problem of projective invariant models and projective invariant estimators.

Another interesting generalization is related to the reconstruction problem for surfaces in 3D: given a set of sample points and normals in 3D, how to describe an affine invariant model of the corresponding surface. Closely related to this question is the definition of good estimators for the affine area and affine curvatures of a surface.



(a) $\kappa = 2.5$ (b) $\kappa = 3.5$ (c) $\kappa = 4.5$

Figure 13: *Influence of the curvature threshold on the reconstruction of a rosace.*

From the practical point of view, the effective use of the estimators in computer vision tasks is certainly a possibility that should be explored.

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References

[1] N. Amenta and Y. J. Kil. Defining point-set surfaces. *Transaction on Graphics*, 23(3):264–270, 2004.

[2] M. Boutin. Numerically invariant signature curves. *International Journal of Computer Vision*, 40(3):235–248, 2000.

[3] S. Buchin. *Affine Differential Geometry*. Academic Press, 1982.

[4] L. D. Cohen. On active contour models and balloons. *Computer Vision and Image Understanding*, 2(53):211–218, 1991.

[5] L. da Fontoura Costa and R. M. Cesar Jr. *Shape analysis and classification*. CRC Press, 2000.

[6] M. Craizer, T. Lewiner and J.-M. Morvan. Parabolic polygons and discrete affine geometry. In *19th Brazilian Symposium on Computer Graphics and Image Processing*, pages 19–26, Manaus, AM, october 2006.

[7] L. H. Figueiredo and J. M. Gomes. Computational morphology of curves. *Visual Computer*, 11:105–112, 1994.

[8] H. Hoppe, T. de Rose, T. Duchamp, J. M. Donald and W. Stuetzle. Surface reconstruction from unorganized points. In *Siggraph*, pages 71–78, 1992.

[9] T. Lewiner, J. Gomes Jr, H. Lopes and M. Craizer. Arc–length based curvature estimator. In *Sibgrapi*, pages 250–257, Curitiba, Oct. 2004. IEEE.

[10] M. Ludwig. Asymptotic approximation by quadratic spline curves. *Annales Universitatis Scientiarum Budapestinensis*, 42:133–139, 1999.

[11] N. J. Mitra and A. T. Nguyen. Estimating surface normals in noisy point cloud data. In *Symposium on Computational Geometry*, pages 322 – 328, 2003.

[12] J. L. Mundy and A. Zisserman, editors. *Geometric invariance in computer vision*. MIT Press, Cambridge, MA, USA, 1992.

[13] E. Calabi, P. J.Olver and A. Tannenbaum. Affine geometry, curve flows, and invariant numerical approximations. *Advances in Mathematics*, 124:154–196, 1997.

[14] E. Calabi, P. J.Olver, C. Shakiban, A. Tannenbaum and S. Hacker. Differential and numerically invariant signature curves applied to object recognition. *International Journal of Computer Vision*, 26(2):107–135, 1998.

[15] B. T. Phong. Illumination for computer generated pictures. *Communication of the ACM*, 18(6):311–317, 1975.

[16] C.-K. Tang, M.-S. Lee and G. Medioni. Tensor Voting. In *Recent Advances in Perceptual Organization*. Kluwer, 2006.

[17] M. Zuliani, S. Bhagavathy, B. S. Manjunath and C. S. Kenney. Affine-invariant curve matching. In *IEEE International Conference on Image Processing*, Oct 2004.

The corresponding work was published in Journal of Mathematical Imaging and Vision, volume 29, numers 2-3, pp. 131–140. Springer, 2006.