Parameterized complexity of discrete Morse theory

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Abstract. Optimal Morse matchings reveal essential structures of cell complexes which lead to powerful tools to study discrete geometrical objects, in particular discrete 3-manifolds. However, such matchings are known to be NP-hard to compute on 3-manifolds, through a reduction to the erasability problem. Here, we refine the study of the complexity of problems related to discrete Morse theory in terms of parameterized complexity. On the one hand we prove that the erasability problem is $W[P]$-complete on the natural parameter. On the other hand we propose an algorithm for computing optimal Morse matchings on triangulations of 3-manifolds which is fixed-parameter tractable in the treewidth of the bipartite graph representing the adjacency of the 1- and 2-simplices. This algorithm also shows fixed parameter tractability for problems such as erasability and maximum alternating cycle-free matching. We further show that these results are also true when the treewidth of the dual graph of the triangulated 3-manifold is bounded. Finally, we discuss the topological significance of the chosen parameters and investigate the respective treewidths of simplicial and generalized triangulations of 3-manifolds.

Keywords: *discrete Morse theory. alternating cycle-free matching. parameterized complexity. fixed parameter tractability. $W[P]$-completeness. treewidth. erasability. collapsibility. computational topology.*
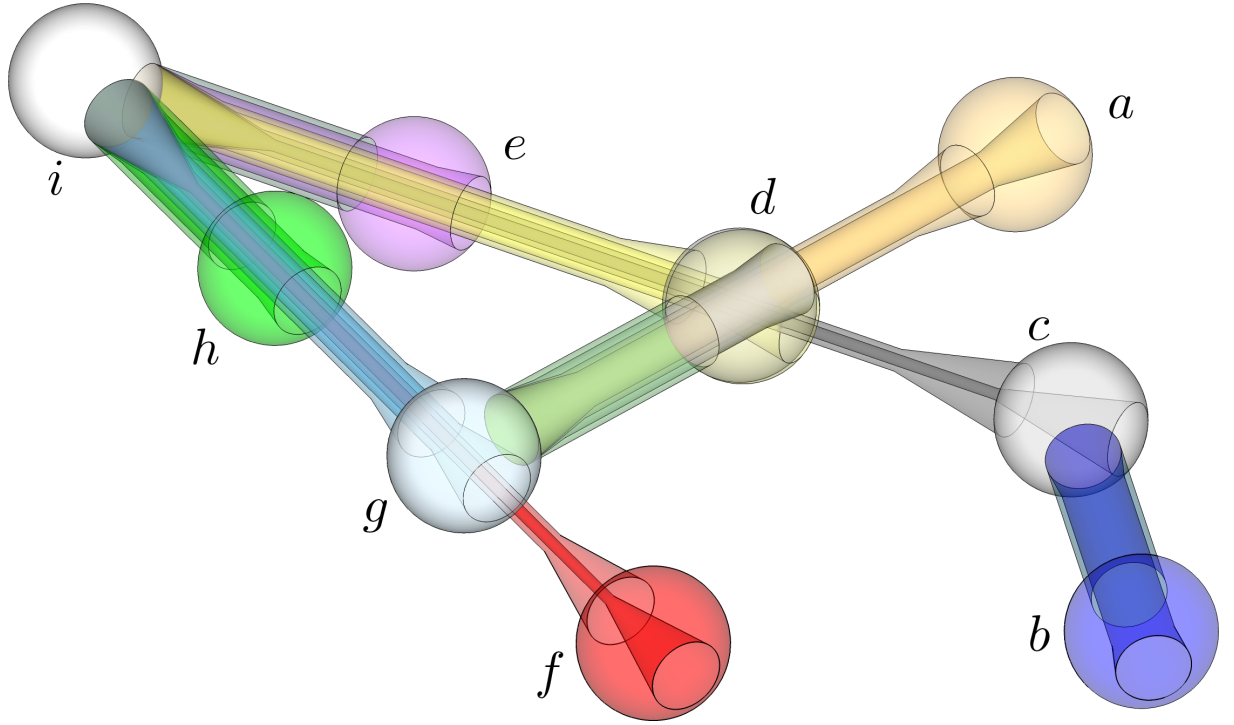
Figure 1: *Constructing an instance of* Erasability *from an instance* (S, R) *of* Minimum Axiom Set*, where* $S = \{a, b, c, d, e, f, g, h, i\}$ *and* $R = \{(\{c, d, e\}, i), (\{f, g, h\}, i), (\{b\}, c), (\{a, d\}, g)\}$.

1 Introduction

Classical Morse theory [52] relates the topology of a manifold to the critical points of scalar functions defined on it, providing efficient tools to understand essential structures on manifolds. Forman [35] recently extended this theory to arbitrary cell complexes. In this discrete version of Morse theory, alternating cycle-free matchings in the Hasse diagram of the cell complex, so-called *Morse matchings*, play the role of smooth functions on the manifold [35, 30]. For example, similarly to the smooth case [54], a closed manifold admitting a Morse matching with only two unmatched (critical) elements is a sphere [35]. The construction of specific Morse matchings has proven to be a powerful tool to understand topological [35, 41, 42, 45, 46], combinatorial [30, 40, 44] and geometrical [38, 48, 15, 56] structures of discrete objects.

Morse matchings that minimize the number of critical elements are known as *optimal matchings* [45]. Together with their number and type of critical elements, these are topological (more precisely homotopy) invariants of the cell complex, just like in the case of the sphere described above. Hence, computing optimal matchings can be used as a purely combinatorial technique in computational topology [4]. Moreover, optimal Morse matchings are useful in practical applications such as volume encoding [47, 59], or homology and persistence computation [48, 37].

However, constructing optimal matchings is known to be NP-hard on general 2-complexes and on 3-manifolds [41, 42, 45]. This result follows from a reduction to this problem from the closely related *erasability problem*: how many faces must be deleted from a 2-dimensional simplicial complex before it can be completely erased, where in each erasing step only *external triangles*, i.e. triangles with an edge not lying in the boundary of any other triangle of the complex, can be removed [34]? Despite this hardness result, large classes of inputs – for which worst case running times suggest the problem is intractable – allow the construction of optimal Morse matchings in a reasonable amount of time using simple heuristics [46, 23]. Such behavior suggests that, while the problem is hard to solve for some instances, it might be much easier to solve for instances which occur in practice. As a consequence, this motivates us to ask what *parameter* of a problem instance is responsible for the intrinsic hardness of the optimal matching problem.

In this article, we study the complexity of discrete Morse theory type problems in terms of parameterized complexity. Following Downey and Fellows [32], an NP-complete problem is called *fixed-parameter tractable* (FPT) with respect to a *parameter* $k \in \mathbb{N}$, if for every input with parameter less than or equal to k, the problem can be solved in $O(f(k) \cdot n^{O(1)})$ time, where f is an arbitrary function independent of the problem size n. For NP-complete but fixed-parameter tractable problems, we can look for classes of inputs for which fast algorithms exist, and identify which aspects of the problem make it difficult to solve. Note that the significance of an FPT result strongly depends on whether the parameter is (i) small for large classes of interesting problem instances and (ii) easy to compute. In addition, the choice of parameter is further limited by classical results in complexity theory. For instance, if the erasability problem for 3-manifolds were FPT in the Betti numbers of the input, this would yield a polynomial time algorithm to decide collapsibility of homology spheres which is unlikely to exist, given the hardness result for arbitrary 3-complexes due to [17].

In order to classify fixed-parameter intractable NP-complete problems, Downey and Fellows [32] propose a family of complexity classes called the W-*hierarchy*:

$$FPT \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq W[P] \subseteq XP.$$

The base problems in each class of the W-hierarchy are versions of satisfiability problems with increasing logical depth as parameter. Problems in $W[P]$ are at most as hard to solve as P-CIRCUIT SAT [32], which asks: given a circuit C of size $p(n)$ with $k \log n$ inputs, fixing k, does C have a satisfying assignment? The rightmost complexity class XP of the W-hierarchy contains all problems which can be solved in $O(n^k)$ time where k is the parameter of the problem.

Here, we use the notion of the W-hierarchy in a geometric setting. More precisely, we determine the hardness of Morse type problems using the mathematically framework of the W-hierarchy. Our first main result shows that the erasability problem is $W[P]$-complete (Theorems 3 and 4), where the parameter is the *natural parameter* – the number of cells that have to be removed. In other words, we prove that the erasability problem is fixed-parameter intractable in this parameter, meaning that there cannot exist an algorithm which runs in time $O(f(k) \cdot n^{O(1)})$ on input with natural parameter $\leq k$, for any computable functions f. From a discrete Morse theory point of view, this reflects the intuition that reaching optimality in Morse matchings requires a global (at least topological) context. In this way, we also show that the W-hierarchy as a purely complexity theoretical tool can be used in a very natural way to answer questions in the field of computational topology. Although there are many results about the computational complexity of topological problems [21, 29, 34, 49, 60], to the authors' knowledge, erasability is the first purely geometric problem shown to be $W[P]$-complete.

Our second main result refines the observation that simple heuristics allow us to compute optimal matchings efficiently. For general 2-complexes (and 3-manifolds), the problem reduces directly to finding a maximum alternating cycle-free matching on a spine, i.e., a bipartite graph representing the 1- and 2-cell adjacencies [22, 42, 47] (Lemma 1). To solve this problem, we propose an explicit algorithm for computing maximum alternating cycle-free matchings which is fixed-parameter tractable in the treewidth of this bipartite graph (Theorem 6), although we did not implement the algorithm yet. Furthermore, we show that finding optimal Morse matchings on triangulated 3-manifolds is also fixed-parameter tractable in the treewidth of the dual graph of the triangulation (Theorem 7).

The treewidth of the dual graph is a common and useful parameter when working with triangulated 3-manifolds [29], as it closely interacts with the topology of the underlying manifold. In order to give further information on the relevance of the fixed parameter results, we will explore this connection in the last section of this article. This will be done by giving explicit examples of large classes of 3-manifolds which admit triangulations of unbounded size with dual graphs of constant tree-width. Moreover, we present experiments using the classification of simplicial and generalized triangulations of 3-manifolds to investigate the "typical" treewidth of the respective graphs for small and relevant instances of Morse type problems. The experiments show that the average treewidths of the respective graphs of simplicial triangulations of 3-manifolds are particularly small in the case of generalized triangulations. Furthermore, experimental data suggest a much more restrictive connection between the treewidth of the dual graph and the spine of triangulated 3-manifolds than the one stated in Theorem 7.

2 Preliminaries

Triangulations

Throughout this paper we mostly consider simplicial complexes of dimensions 2 and 3, although most of our results hold for more general combinatorial structures. All 2-dimensional simplicial complexes we consider are (i) pure, i.e., all maximal simplices are triangles (2-simplices) and (ii) *strongly connected*, i.e., each pair of triangles is connected by a path of triangles such that any two consecutive triangles are joined by an edge (1-simplex). The 0-simplices of the complex are also called *vertices*. All 3-dimensional simplicial complexes we consider are triangulations of closed 3-manifolds, that is, simplicial complexes whose underlying topological space is a closed 3-manifold. In particular every 3-manifold can be represented in this way [51]. We will refer to these objects as *simplicial triangulations of 3-manifolds*.

In §5 we briefly concentrate on a slightly more general notion of a *generalized triangulation of a 3-manifold*, which is a collection of tetrahedra all of whose faces are affinely identified or "glued together" such that the underlying topological space is a 3-manifold. Generalized triangulations use far fewer tetrahedra than simplicial complexes, which makes them important in computational 3-manifold topology (where many algorithms run exponential time). Every simplicial triangulation is a generalized triangulation, and the second barycentric subdivision of a generalized triangulation is a simplicial triangulation [51], hence both objects are closely related.

For the remainder of this article, we will often consider 2-dimensional simplicial complexes as part of a simplicial triangulation of a 3-manifold.

Erasability of simplicial complexes

Let Δ be a 2-dimensional simplicial complex. A triangle $t \in \Delta$ is called *external* if t has at least one edge which is not in the boundary of any other triangle in Δ; otherwise t is called *internal*. Given a 2-dimensional simplicial complex Δ and a triangle $t \in \Delta$, the 2-dimensional simplicial complex obtained by removing (or *erasing*) t from Δ is denoted by $\Delta \setminus t$. In addition, if Δ' is obtained from Δ by iteratively erasing triangles such that in each step the erased triangle is external in the respective complex, we will write $\Delta \rightsquigarrow \Delta'$. We say that the complex Δ is *erasable* if $\Delta \rightsquigarrow \delta$, where δ denotes a subcomplex of Δ with no triangle. Finally, for every 2-dimensional simplicial complex Δ we define $\mathrm{er}(\Delta)$ to be the smallest size of a subset Δ_0 of triangles of Δ such that $\Delta \setminus \Delta_0 \rightsquigarrow \delta$. The elements of Δ_0 are called *critical* triangles and hence $\mathrm{er}(\Delta)$ is sometimes also referred to as the *minimum number of critical triangles* of Δ. Determining $\mathrm{er}(\Delta)$ is known as the erasability problem [34].

Problem 1 (ERASABILITY).

INSTANCE   A 2-dimensional simplicial complex Δ.

PARAMETER   A non-negative integer k.

QUESTION   Is $\mathrm{er}(\Delta) \leq k$?

Hasse diagram and spine

Note: we will refer to the vertices and edges of the Hasse diagram – and other graphs in this paper – as *nodes* and *arcs* to avoid confusion with the vertices and edges of a triangulation.

Given a simplicial complex Δ, one defines its *Hasse diagram H* to be a directed graph in which the set of nodes of H is the set of simplices of Δ, and an arc goes from τ to σ if and only if σ is contained in τ and $\dim(\sigma) + 1 = \dim(\tau)$. Let $H_i \subseteq H$ be the bipartite subgraph spanned by all nodes of H corresponding to i- and $i + 1$-dimensional simplices. In particular, H_1 describes the adjacency between the 2-simplices and 1-simplices of Δ, and will be called the *spine* of the simplicial complex Δ. The spine of a simplicial complex will be one of the main objects of study in this work.

Matchings

By a *matching of a graph $G = (N, A)$* we mean a subset of arcs $M \subset A$ such that every node of N is contained in at most one arc in M. Arcs in M are called *matched arcs* and the nodes of the matched arcs are called *matched nodes*. Nodes and arcs which are not matched are referred to as *unmatched*. The *induced M-subgraph* is the subgraph of G spanned by all matched nodes and the *size* of a matching M is the number of matched arcs. A matching M is called a *maximum matching* of a graph G if there is no matching with a larger size than the size of M.

Morse matchings

Let H be the Hasse diagram of a simplicial complex Δ and M be a matching on H. Let $H(M)$ be the directed graph obtained from the Hasse diagram by reversing the direction of each arc of the matching M. If $H(M)$ is a directed acyclic graph, i.e., $H(M)$ does not contain directed cycles, then M is a *Morse matching* [30]. Furthermore, the number c_i of unmatched nodes representing i-simplices of Δ is called the number of *critical i-dimensional simplices* and the sum $c(M) = \sum_i c_i$ is said to be the *total number of critical simplices*.

The motivation to find optimal Morse matchings is given by the following fundamental theorem of discrete Morse theory due to Forman which deals with simple homotopy reductions to CW-complexes [3, 10].

Theorem 1 ([35])**.** *Let M be a Morse matching on a simplicial complex Δ. Then Δ is homotopy equivalent to a CW-complex with exactly one d-cell for each critical d-simplex of M.*

In other words, a Morse matching with the smallest number of critical simplices gives us the most compact and succinct topological representation up to homotopy. For more information about the basic facts of Morse theory we refer the reader to Forman's original work [35]. This motivates a fundamental problem in discrete Morse theory, *optimal Morse matching*, as a decision problem in the following form.

Problem 2 (MORSE MATCHING)**.**

INSTANCE   A simplicial complex Δ.
PARAMETER   A non-negative integer k.
QUESTION   Is there a Morse matching M with $c(M) \leq k$?

Note that ERASABILITY can be restated as a version of MORSE MATCHING where only the number of unmatched 2-simplices (that is, $c_2(M)$) is counted [45].

Complexity of Morse matchings

The complexity of computing optimal Morse matchings is linear on 1-complexes (graphs) [35] and 2-complexes that are manifolds [45]. Joswig and Pfetsch [42] prove that if you can solve ERASABILITY in the spine of a 2-simplicial complex in polynomial time, then you can solve MORSE MATCHING in the *entire* complex in polynomial time. The proof technique easily extends to 3-manifolds, leading to the following lemma which has been mentioned in previous works [47, 42, 22].

Lemma 1. *Let M be a Morse matching on a triangulated 3-manifold Δ. Then we can compute a Morse matching M' in polynomial time which has exactly one critical 0-simplex and one critical 3-simplex, such that $c(M') \leq c(M)$.*

In other words, answering ERASABILITY on the spine is the only difficult part when solving MORSE MATCHING on a 3-manifold. In §4 we show that if a spine has bounded treewidth, then we can solve ERASABILITY in linear time. Lemma 1 therefore generalizes this result to MORSE MATCHING on 3-manifolds.

The proof of Lemma 1 actually follows directly from Joswig and Pfetsch's proof of the following lemma.

Lemma 2 ([42])**.** *Let M be a Morse matching on a 2-simplicial complex Δ. Then we can compute a Morse matching M' in polynomial time which has exactly one critical 0-simplex, the same number of critical simplices of dimension greater than or equal 2 as M, and $c(M') \leq c(M)$.*

The proof builds a Morse matching from a spanning tree of the *primal graph*, i.e. the graph obtained considering only the vertices and edges of Δ. For a 3-manifold Δ, the proof of the previous lemma can be applied exactly the same way on the *dual graph*, i.e. the graph whose nodes represent tetrahedra of Δ, and whose arcs represent common triangles of Δ joined together (cf. Definition 5), to obtain the following result.

Lemma 3. *Let M be a Morse matching on a closed triangulated 3-manifold Δ. Then we can compute a Morse matching M' in polynomial time which has exactly one critical 3-simplex, the same number of critical simplices of dimension less than or equal to 1, and $c(M') \leq c(M)$.*

Since the proof works independently on the primal and dual graph, Lemma 1 is a combination of these results. Here, we simply reproduce the proof of Joswig and Pfetsch [42] verbatim applying it to 3-manifold complexes, using Poincaré's duality.

First consider a Morse matching M for a connected 3-manifold Δ. Let $\gamma(M)$ be obtained from the primal graph of Δ by removing all arcs (edges of Δ) matched with triangles and let $\gamma^*(M)$ be derived from the dual graph of Δ by removing all arcs corresponding to triangles matched with tetrahedra of Δ. Note that $\gamma(M)$ contains all vertices (0-simplices) and $\gamma^*(M)$ contains all tetrahedra of Δ as nodes.

Lemma 4. *The graphs $\gamma(M)$ and $\gamma^*(M)$ are connected.*

Proof. Suppose that $\gamma(M)$ is disconnected. Let N be the set of nodes in a connected component of $\gamma(M)$, and let C be the set of cut edges, that is, edges of Δ with one vertex in N and one vertex in its complement. Since Δ is connected, C is not empty. By definition of $\gamma(M)$, each edge in C is matched to a unique 2-simplex.

Consider the directed subgraph D of the Hasse diagram consisting of the edges in C and their matching 2-simplices. The standard direction of arcs in the Hasse diagram (from the higher to the lower dimensional simplices) is reversed for each matching pair of M, i.e., D is a subgraph of $H(M)$. We construct a directed path in D as follows. Start with any node of D corresponding to a cut edge e_1 and consider the node of D determined by the unique 2-simplex τ_1 matched with e_1. Then τ_1 contains at least one other cut edge e_2, otherwise e_1 cannot be a cut edge. Now iteratively go to e_2, then to its unique matching 2-simplex τ_2, choose another cut edge e_3, and so on. We observe that we obtain a directed path $e_1, \tau_1, e_2, \tau_2, \cdots$ in D, i.e., the arcs are directed in the correct direction. Since we have a finite graph at some point the path must arrive at a node of D which we have visited already. Hence, D (and therefore also $H(M)$) contains a directed cycle, which is a contradiction since M is a Morse matching.

To prove that $\gamma^*(M)$ is connected, we repeat the proof above on the dual graph. $\square$

Lemma 1. Since $\gamma(M)$ and $\gamma^*(M)$ are connected, they both have spanning trees, and we will use them to build the Morse matching. First pick an arbitrary node r_1 and any spanning tree of $\gamma(M)$ and match every other node (vertex of Δ) of the spanning tree to the arc (edge of Δ) that points towards r_1. Then pick an arbitrary tetrahedron (node in the dual graph) r_2 and any spanning tree of $\gamma^*(M)$ and match every other node of the spanning tree to the arc (triangle of Δ) that points towards r_2. This yields a maximum Morse matching on $\gamma(M)$ and $\gamma^*(M)$. It is easy to see that replacing the part of M on $\gamma(M)$ and $\gamma^*(M)$ with this matching yields a Morse matching. This Morse matching has only one critical vertex (the root r_1) and one critical tetrahedron (the root r_2). Note that every Morse matching in a triangulated 3-manifold contains at least one critical vertex and at least one critical tetrahedron; this can be seen from Theorem 1. Furthermore, the total number of critical simplices can only decrease, since we computed an optimal Morse matching on $\gamma(M)$ and $\gamma^*(M)$. $\square$

3 W[P]-Completeness of the ERASABILITY problem

In order to prove that ERASABILITY is $W[P]$-complete in the natural parameter, we first have to take a closer look at what has to be considered when proving hardness results with respect to a particular parameter.

Definition 1 (Parameterized reduction)**.** *A parameterized problem L reduces to a parameterized problem L', denoted by $L \leq_{FPT} L'$, if we can transform an instance (x, k) of L into an instance $(x', g(k))$ of L' in time $f(k)|x|^{O(1)}$ (where f and g are arbitrary functions, and $|x|$ means the size of x), such that (x, k) is a yes-instance of L if and only if $(x', g(k))$ is a yes-instance of L'.*

As an example, Eğecioğlu and Gonzalez [34] reduce SET COVER to ERASABILITY to show that ERASABILITY is NP-complete. Since their reduction approach turns out to be a parameterized reduction, these results can be restated in the language of parameterized complexity as follows.

Corollary 1. SET COVER $\leq_{FPT}$ ERASABILITY*, therefore* ERASABILITY *is $W[2]$-hard.*

This shows that, if the parameter k is simultaneously bounded in both problems, ERASABILITY is *at least as hard as* SET COVER. Eğecioğlu and Gonzalez [34] conjectured that ERASABILITY is *harder* than SET COVER. In this section we will prove this conjecture and determine exactly how much harder ERASABILITY is than SET COVER, which is $W[2]$-complete.

Namely, we will show that ERASABILITY is $W[P]$-complete in the natural parameter k. This will be done by i) using a $W[P]$-complete problem as an oracle to solve an arbitrary instance of ERASABILITY (Theorem 3, which shows that ERASABILITY is in $W[P]$), and ii) reducing an arbitrary instance of a suitable problem which is known to be $W[P]$-complete to an instance of ERASABILITY (Theorem 4, which shows that ERASABILITY is $W[P]$-hard). Moreover, since our proof involves only a local replacement that does not use the natural parameter k, ERASABILITY is parametrically inapproximable, as well [7, 12] (see [6, Chapter 31] for more about parameterized approximations).

There are only a few problems described in the literature which are known to be $W[P]$-complete [33, p. 473]. Amongst these problems, the following is suitable for our purposes.

Problem 3 (MINIMUM AXIOM SET).

INSTANCE   A finite set S of *sentences*, and an *implication relation* R consisting of pairs (U, s) where $U \subseteq S$ and $s \in S$.

PARAMETER   A positive integer k.

QUESTION   Is there a set $S_0 \subseteq S$ (called an *axiom set*) with $|S_0| \leq k$ and a positive integer n, for which $S_n = S$, where we define S_i, $1 \leq i \leq n$, to consist of exactly those $s \in S$ for which either $s \in S_{i-1}$ or there exists a set $U \subseteq S_{i-1}$ such that $(U, s) \in R$?

Theorem 2 ([32]). MINIMUM AXIOM SET *is $W[P]$-complete.*

In this paper, we show that, preserving the natural parameter k, MINIMUM AXIOM SET is both at least and at most as hard as ERASABILITY.

Theorem 3. ERASABILITY $\leq_{FPT}$ MINIMUM AXIOM SET, *therefore* ERASABILITY *is in $W[P]$.*

Proof. We show membership of ERASABILITY in $W[P]$ by reducing a given instance (Δ, k) of ERASABILITY to an instance (S, R, k) of MINIMUM AXIOM SET.

W. l. o. g. we can assume that the 2-dimensional simplicial complex Δ has no external edges (if Δ has external edges we first remove these edges until no external edge exists and reduce the remaining problem instance to an instance of MINIMUM AXIOM SET). We now identify the set of triangles of Δ with the set of sentences S in a one-to-one correspondence. For every edge $e \in \Delta$ we denote the set of all triangles containing e by $\operatorname{star}_\Delta(e) \subset \Delta$, we write for the corresponding set of sentences $S_e \subset S$, and we define the set of implication relations R by the relations

$$(S_e \setminus \{s\}, s)$$

for each triangle $s \in S_e$ for all edges $e \in \Delta$. Note that Δ has no external edges and thus $S_e \setminus \{s\} \neq \emptyset$ for all e.

In a next step, we show that for all axiom sets $S_0 \subset S$ of size k we have $\Delta \setminus \Delta_0 \rightsquigarrow \delta$, δ 1-dimensional subcomplex of Δ, for the associated subset of triangles $\Delta_0 \subset \Delta$ of size k. To see that this is true, note that for the augmenting sequence $S_0 \subset S_1 \subset \ldots \subset S_n = S$ of S, their corresponding subsets of triangles $\Delta_0 \subset \Delta_1 \subset \ldots \subset \Delta_n = \Delta$ and $i \in \{1, \ldots, n\}$ fixed, all sentences $s \in S_i \setminus S_{i-1}$ have to occur in a relation $(S_e \setminus \{s\}, s)$ for some edge e with $S_e \setminus \{s\} \subset S_{i-1}$. For the triangle $t \in \Delta$ corresponding to s this means that, $\operatorname{star}_\Delta(e) \setminus t \subset \Delta_{i-1}$. Thus, if we assume that all triangles in Δ_{i-1} are already erased, t must be external and thus can be erased as well. The statement now follows by the fact that for $i = 1$, all triangles in Δ_0 are already erased in $\Delta \setminus \Delta_0$ and hence $\Delta \setminus \Delta_0 \rightsquigarrow \delta$.

Conversely, let $\Delta_0 \subset \Delta$ be of size k such that $\Delta \setminus \Delta_0 \rightsquigarrow \delta$. Since Δ has no external triangles but $\Delta \setminus \Delta_0 \rightsquigarrow \delta$, there must be external triangles $t \in \Delta \setminus \Delta_0$ and hence for $s \in S$ being the sentence corresponding to the triangle t there is a relation $(S_e \setminus \{s\}, s)$ with $S_e \setminus \{s\} \subset S_0$, where S_0 is the set of sentences corresponding to the set of triangles Δ_0. We then define S_1 to be the union of S_0 with all sentences s of the type described above and iterating this step results in a sequence of subsets $S_0 \subset S_1 \subset \ldots \subset S_n = S$ for some n what proves the result. □

In order to show that it is in fact amongst the hardest problems in this class we first need to build some gadgets.

Definition 2 (Gadgets for the hardness proof of ERASABILITY). *Let (S, R, k) be an instance of* MINIMUM AXIOM SET.

Let $s \in S$ be a sentence. By an s-gadget or sentence gadget we mean a triangulated 2-dimensional sphere with $2n + m$ punctures as shown in Figure 2, where m is the number of relations $(U, s) \in R$ and n is the number of relations $(U, s) \in R$ such that $s \in U$.

Let $(U, s) \in R$ be a relation. A (U, s)-gadget or implication gadget is a collection of $|U| + 1$ sentence gadgets for each sentence of $U \cup \{s\}$ together with $2|U|$ nested tubes as shown in Figure 3 such that (i) two tubes are attached to two punctures of the u-gadget for each $u \in U$ and (ii) all $2|U|$ boundary components at the other side of the tubes are identified at a single puncture of the s-gadget.
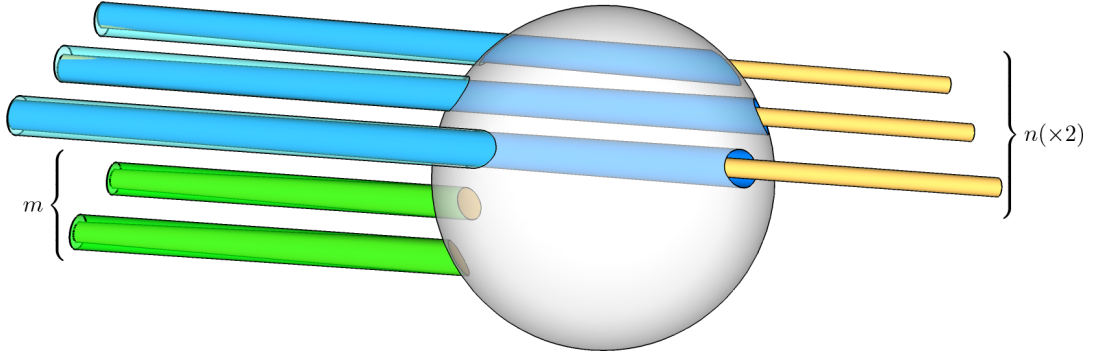
Figure 2: *Example of a sentence gadget with $m = 2$ relations (U, s) and $n = 3$ relations (U, u) with additional tubes.*
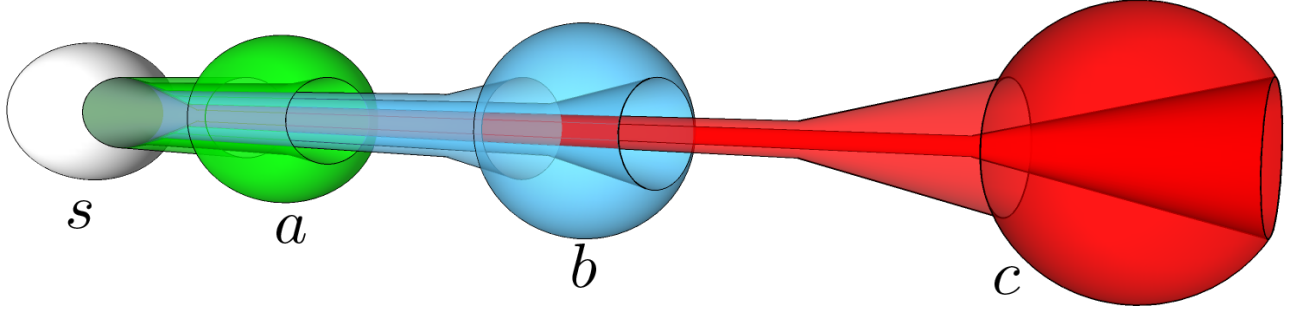
Figure 3: *Example of a (U, s)-gadget with $U = \{a, b, c\}$, with sentence gadgets $\{a, b, c, s\}$.*

Then, by construction the following holds for the (U, s)-gadget.

Lemma 5. *A (U, s)-gadget can be erased if and only if all sentence gadgets corresponding to sentences in U are already erased.*

Proof. Clearly, if all sentence gadgets corresponding to sentences in U are erased, the whole gadget can be erased tube by tube. If, on the other hand, one of the sentence gadgets still exists, this gadget together with the two tubes connected to it build a complex without external triangles which thus cannot be erased. □

With these tools in mind we can now prove the main theorem of this section.

Theorem 4. MINIMUM AXIOM SET $\leq_{FPT}$ ERASABILITY, ERASABILITY *is $W[P]$-hard even when the instance of* ERASABILITY *is a strongly connected pure 2-dimensional simplicial complex Δ which is embeddable in $\mathbb{R}^3$. Therefore* ERASABILITY *is $W[P]$-hard.*

The simplicial complex Δ (Figure 1) constructed to prove $W[P]$-hardness of ERASABILITY is in fact embeddable into $\mathbb{R}^3$. This means that, even in the relatively well-behaved class of embeddable 2-dimensional simplicial complexes, ERASABILITY when bounding the number of critical simplices is still likely to be inherently difficult.

Proof. To show $W[P]$-hardness of ERASABILITY, we will reduce an arbitrary instance (S, R, k) from MINIMUM AXIOM SET to an instance (Δ, k) of ERASABILITY. In order to do so, we will use a sentence gadget for each element of S and an implication gadget for each relation R (cf. Definition 2) to construct a 2-dimensional simplicial complex Δ with a polynomial number of triangles in the input size.

By construction, we can glue all sentence and implication gadgets together in order to obtain a simplicial complex Δ without any exterior triangles. Note that the only place where Δ is not a surface is at the former m boundary components of the sentence gadgets corresponding to the right hand sides of the relations in R.

For any axiom set $S_0 \subset S$ of size k, let Δ_0 be a set of k triangles, one from each sentence gadget corresponding to a sentence in S_0. It follows by Lemma 5, that $\Delta \setminus \Delta_0$ can be erased to a complex where all the sentence gadgets s corresponding to relations (U, s), $U \subset S_0$, have external triangles. Since S_0 is an axiom set, iterating this process erases the whole complex Δ.

Conversely, let Δ_0 be a set of k triangles such that $\Delta \setminus \Delta_0 \rightsquigarrow \delta$. First, note that erasing a triangle of any tube of an implication gadget always allows us to remove the sentence gadget at the right end of this tube. Hence, w. l. o. g. we can assume that all k

triangles in Δ_0 are triangles of some sentence gadget in Δ. Now, if any sentence gadget contains more than one triangle of Δ_0 we delete all additional triangles obtaining a set Δ'_0 of k' triangles, $k' \leq k$, such that $\Delta \setminus \Delta'_0 \rightsquigarrow \delta$ and thus the corresponding set of sentences is an axiom set of size $k' \leq k$.

The result now follows by the observation that Δ can be realized by at most a quadratic number of triangles in the input size of (S, R, k). □

The $W[P]$-completeness result implies that if ERASABILITY turns out to be fixed parameter tractable, then $W[P] = FPT$, i.e., every problem in $W[P]$ including the ones lower in the hierarchy would turn out to be fixed parameter tractable, an unlikely and unexpected collapse in parameterized complexity. Also, it would imply that the n-variable SAT problem could be solved in time $2^{o(n)}$, that is, better than a brute force search [20]. With respect to this result, if we want to prove fixed parameter tractability of ERASABILITY, the parameter must be different from the natural parameter.

4 Fixed parameter tractability in the treewidth

In this section, we prove that there is still hope to find an efficient algorithm to solve MORSE MATCHING. We give positive results for the field of discrete Morse theory by proving that ERASABILITY and MORSE MATCHING are fixed parameter tractable in the treewidth of the spine of the input simplicial complex, and also in the dual graph of the problem instance in case it is a simplicial triangulation of a 3-manifold.
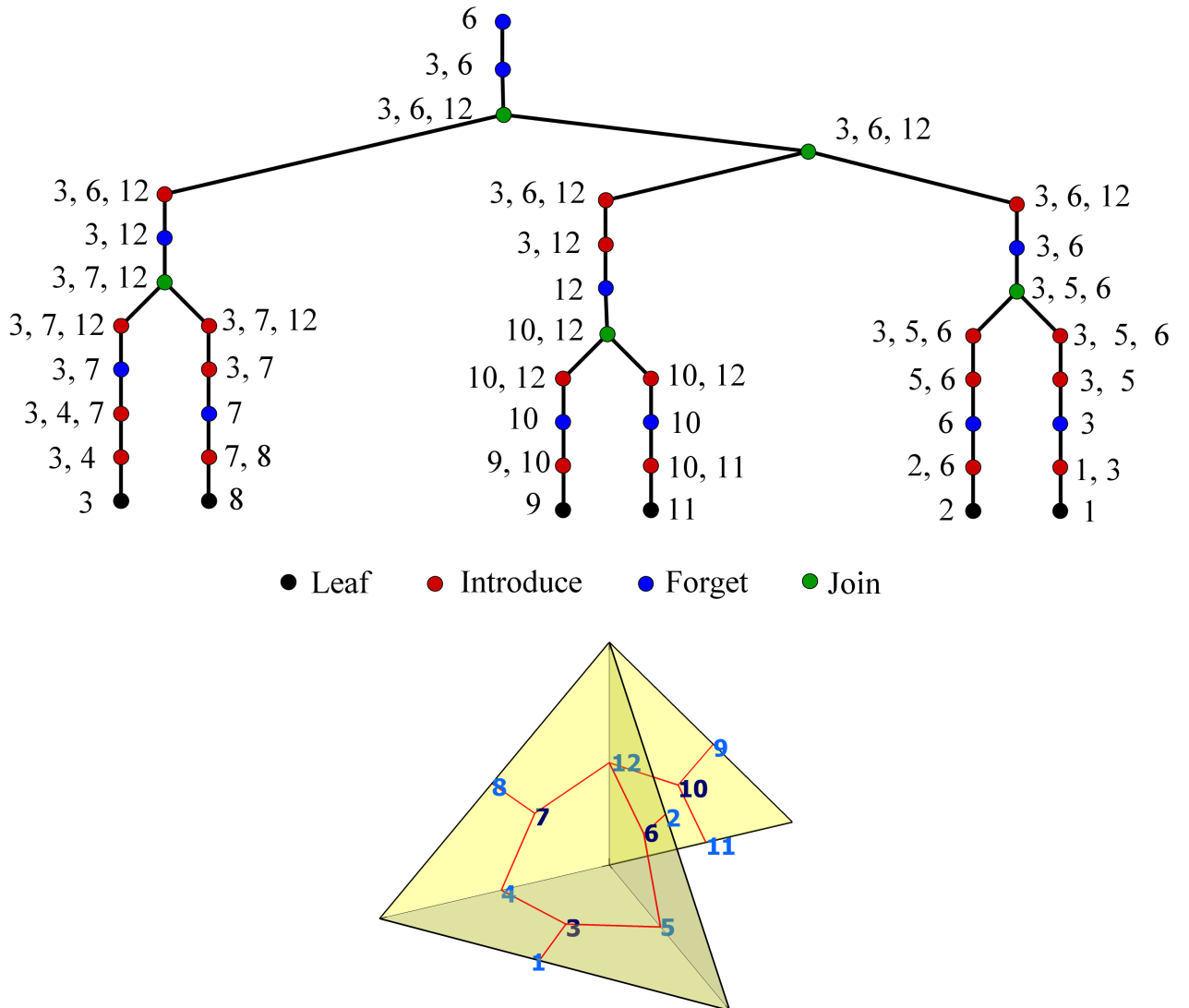
Figure 4: *Example of a nice tree decomposition (left) of the spine of a non-manifold 2-dimensional simplicial complex (right).*

(a) Treewidth

Definition 3 (Treewidth). *A tree decomposition of a graph G is a tree T whose nodes $\{X_i \mid i \in I\}$ are called* bags*. Each bag X_i is a subset of nodes of G, and we require that:*

> [leftmargin=2cm]

node coverage: *every node of G is contained in at least one bag X_i;*

arc coverage: *for each arc of G, some bag X_i contains both its endpoints;*

coherence: *for all bags X_i, X_j and X_k of T, if X_j lies on the unique simple path from X_i to X_k in T, then $X_i \cap X_k \subseteq X_j$.*

The width of a tree decomposition is defined as $\max |X_i| - 1$, and the treewidth of G is the minimum width over all tree decompositions. We will denote the treewidth of G by $\mathbf{tw}(G)$.

For bounded treewidth, computing a tree decomposition of a graph $G = (V, E)$ of width $\leq k$ has running time $O(f(k)|V|)$ due to an algorithm by Bodlaender [25]. Regarding the size of $f(k)$: using the improved algorithm by Perković and Reed [14], at most $O(k^2)$ recursive calls of Bodlaender's improved linear time fixed-parameter tractable algorithm for bounded treewidth [1] are needed. This latter algorithm in turn is said to have a constant factor $f(k)$ which is "at most singly-exponential in k". Details on the running times of tree decomposition algorithms are available in the literature [2, 43].

Definition 4 (Nice tree decomposition). *A tree decomposition $(\{X_i \mid i \in I\}, T)$ is called a* nice tree decomposition *if the following conditions are satisfied:*

1. *There is a fixed bag X_r with $|X_r| = 1$ acting as the root of T (in this case X_r is called the* root bag*).*

2. *If bag X_j has no children, then $|X_j| = 1$ (in this case X_j is called a* leaf bag*).*

3. *Every bag of the tree T has at most two children.*

4. *If a bag X_i has two children X_j and X_k, then $X_i = X_j = X_k$ (in this case X_i is called a* join bag*).*

5. *If a bag X_i has one child X_j, then either*

   (a) *$|X_i| = |X_j| + 1$ and $X_j \subset X_i$ (in this case X_i is called an* introduce bag*), or*

   (b) *$|X_j| = |X_i| + 1$ and $X_i \subset X_j$ (in this case X_i is called a* forget bag*).*

A given tree decomposition can be transformed into a nice tree decomposition (Figure 4) in linear time:

Lemma 6 ([43]). *Given a tree decomposition of a graph G of width w and $O(n)$ bags, where n is the number of nodes of G, we can find a nice tree decomposition of G that also has width w and $O(n)$ bags in time $O(n)$.*

(b) Alternating cycle-free matchings

Given a graph $G = (N, A)$ and a matching $M \subset A$ on G, an *alternating path* is a sequence of pairwise adjacent arcs such that each matched arc in the sequence is followed by an unmatched arc and conversely. An *alternating cycle* of M is a closed alternating path. Matchings which do not have any such alternating cycle are called *alternating cycle-free matchings*. From the definition of Morse matching, we can state ERASABILITY in the language of alternating cycle-free matchings as follows:

Problem 4 (ALTERNATING CYCLE-FREE MATCHING).

INSTANCE   A bipartite graph $G = (N_1 \cup N_2, A)$.

PARAMETER   A nonnegative integer k.

QUESTION   Does G have an alternating cycle-free matching M with at most k unmatched nodes in N_1?

Specifically, if $G = H_1$ is the spine for some simplicial complex Δ, then ERASABILITY is equivalent to the ALTERNATING CYCLE-FREE MATCHING problem.

(c) FPT algorithm for the alternating cycle-free matching problem

Courcelle's theorem [31] can be used to show that ALTERNATING CYCLE-FREE MATCHING is fixed parameter tractable. Indeed, Courcelle's celebrated theorem [31] states that all graph properties that can be defined in *Monadic Second-Order Logic* (MSOL) can be decided in linear time when the graph has bounded treewidth. Here, we want to use Courcelle's theorem to show that problems in discrete Morse theory are fixed parameter tractable in the treewidth of some graph associated to the problem. However, it is not obvious how to *directly* state ERASABILITY and MORSE MATCHING in MSOL. Instead, we will apply Courcelle's theorem to ALTERNATING CYCLE-FREE MATCHING.

Theorem 5. *Let $w \geq 1$. Given a bipartite graph with $\mathbf{tw}(G) \leq w$,* ALTERNATING CYCLE-FREE MATCHING *can be solved in linear time.*

Proof. Let $G = (N_1 \cup N_2, A)$ be a bipartite graph and let $N = N_1 \cup N_2$ be the set of nodes of G. We will write an MSOL formulation of ALTERNATING CYCLE-FREE MATCHING based on the fact that $M \subset N$ is an alternating cycle-free matching if and only if M is a matching and every induced M-subgraph contains a node of degree 1 [36]:

$$\max M : \forall x \in N[\neg \exists a_1, a_2 \in M(a_1 \neq a_2 \wedge inc(x, a_1) \wedge inc(x, a_2))]$$
$$\wedge \, \forall M' \subseteq M(\exists a \in M', \exists x \in N[inc(x, a) \wedge (\forall x_1 \in N,$$
$$(\neg \exists a_1 \in M'(x \neq x' \wedge adj(x, x_1) \wedge inc(x_1, a_1))))]),$$

where $inc(x, a)$ is the incidence predicate between node x and arc a and $adj(x, x')$ is the adjacency predicate between node x and node x'. The above statement can be translated to plain English as follows: "Find the largest matching M of G, where each node is incident to at most one arc, such that in every subset M' of the matching M there exists a matched node x in M' such that its only neighbor matched in M' is the other endpoint of the unique matched arc incident to x. $\qquad\square$

However, this is a purely theoretical result, since the stated complexity contains towers of exponents in the parameter function. This is the reason why, for the remainder of this section, we focus on the construction of a linear time algorithm to solve ALTERNATING CYCLE-FREE MATCHING for inputs of bounded treewidth with a significantly faster running time.

Theorem 6. *Let $G = (N_1 \cup N_2, A)$ be a simple bipartite graph with a given nice tree decomposition $(\{X_i \,|\, i \in I\}, T)$. Then the size of a maximum alternating cycle-free matching of G can be computed in $O(4^{w^2+w} \cdot w^3 \cdot \log(w) \cdot n)$ time, where $n = |N|$ and w denotes the width of the tree decomposition.*

Algorithm overview

Our algorithm constructs alternating cycle-free matchings of G along the nice tree decomposition $(\{X_i \,|\, i \in I\}, T)$ of G, from the leaves up to the root, visiting each bag exactly once. In the following we will denote by F_i, the set of nodes which are already processed and forgotten by the time X_i is reached; we call this the *set of forgotten nodes*. At each bag X_i of the decomposition, we construct a set $\mathcal{M}(i)$ representing *all* valid alternating cycle-free matchings in the graph induced by the nodes in $X_i \cup F_i$.

The leaf bags contain a single node of G, and the only matching is thus empty. At each introduce bag $X_i = X_j \cup \{x\}$, each matching M of $\mathcal{M}(j)$ can be extended to several matchings as follows. The newly introduced node x can be either left unmatched, or matched with one of its neighbors as long as it generates a valid and cycle-free matching with M. At each join bag $X_i = X_j = X_k$, $\mathcal{M}(i)$ is built from the valid combinations of pairs of matchings from $\mathcal{M}(j)$ and $\mathcal{M}(k)$. The final list of valid matchings is then evaluated at the root bag r.

However, this final list $\mathcal{M}(r)$ contains an exponential number of matchings. Fortunately, the nice tree decomposition allows us to group together, at each step, all matchings M that coincide on the nodes of X_i. Indeed, the algorithm takes the same decisions for all the matchings of the group. We can thus store and process a much smaller list $\mathcal{M}(j)$ of matchings containing only one representative $\tilde{M}$ of each group. In each group, we choose one with the smallest number of unmatched nodes so far. This grouping takes place at the forget and join bags. This makes the algorithm exponential in the bag size, not the input size.

Matching data structure

The structure storing an alternating cycle-free matching M in a set $\mathcal{M}(i)$ must be suitable for checking the matching validity whenever a matching is extended at an introduce bag or a join bag. It must store which nodes are already matched in M to avoid matching a node of G twice (*matching condition*). We use a binary vector $\mathbf{v}(M)$, where the x-th coordinate is 1 if node $x \in X_i$ is matched and 0 otherwise. Checking the matching condition and updating when nodes are matched has thus a constant execution time $O(1)$.

Also, the structure must store which nodes are connected by an alternating path in M to avoid closing a cycle when extending or combining M (*cycle-free condition*). When matching two nodes x and y, an alternating cycle is created if there exists an alternating path from a neighbor of x to a neighbor of y. To test this, we use a union-find structure [18] $\mathbf{uf}(M)$, storing for each matched node x the index of a matched node $c(x)$ connected to x by an alternating path in M. For a subset of matched nodes which are all connected to each other, the component index c is chosen to be the node with the lowest index. For each unmatched node, we store the ordered list of component indices of *matched* neighbor nodes. The cycle-free condition check reduces to find calls on the adjacent lists, and the update of the structure when increasing the matching size reduces to union calls, both executing in near-constant time (We actually have to add a near constant factor of the Ackermann function. However, this will be dominated by the complexity of sorting $\mathbf{v}$). All the matchings are stored in a hash structure to allow faster search for duplicates. Finally, we can return not only the maximum cycle-free matching size, but an actual maximum cycle-free matching by storing, along with each representative matching, a binary vector of size $|X_i \cup F_i|$ with all the matched nodes so far.

Grouping

Traversing the nice tree decomposition in a bottom-up fashion, each node appears in a set of bags that form a subtree of the tree decomposition (*coherence requirement*). This means that, whenever a node is forgotten, it is never introduced again in the bottom-up traversal.

A naïve version of the algorithm described above would build the complete list of valid alternating cycle-free matchings: the set $\mathcal{M}(i)$ would contain all valid matchings in the graph induced by the nodes in $X_i \cup F_i$. In particular, for each matching $M \in \mathcal{M}(i)$ the algorithm would store the binary vector $\mathbf{v}(M)$ and the union-find structure $\mathbf{uf}(M)$ on $X_i \cup F_i$. However, it is sufficient to store the essential information about each M by restricting the union-find structure $\mathbf{uf}(M)$ and the binary vector $\mathbf{v}(M)$ *only* to the nodes in the bag X_i (for any matched node $x \in X_i$, node $c(x)$ of the union-find structure is then chosen inside X_i). More precisely, we define an equivalence relation $\sim_i$ on the matchings of $\mathcal{M}(i)$ such that $M \sim_i M'$ if and only if $\mathbf{v}(M) = \mathbf{v}(M')$ and $\mathbf{uf}(M) = \mathbf{uf}(M')$ on the nodes of X_i. Since two equivalent matchings only differ on the forgotten nodes F_i, the validation of the matching and cycle-free conditions of any extension of M or M' (or any combination with a third equivalent matching M'') will be equal from now on.

Since we are interested in the alternating cycle-free matching with the minimum number of unmatched nodes, for each equivalence class we will choose a matching $\tilde{M}$ with the minimum number $m(\tilde{M})$ of unmatched forgotten nodes as class representative. This number $m(\tilde{M})$ is stored together with $(\mathbf{v}, \mathbf{uf})$ for each equivalence class of $\tilde{\mathcal{M}}(i) = \mathcal{M}(i)/\sim_i$. In addition, we can compute the alternating cycle-free matching of maximum size by storing the complete binary vector $\mathbf{v}$ along with $m(\tilde{M})$ (since the matching is cycle-free, this is sufficient to recover the set of arcs defining the matching).

Execution time complexity

To measure the running time we need to bound the number of equivalence classes of $\tilde{\mathcal{M}}(i)$. Let w_i be the number of nodes in X_i. The number of equivalence classes of $\tilde{\mathcal{M}}(i)$ is then bounded above by the number of possible pairs $(\mathbf{v}, \mathbf{uf})$ on w_i nodes. The union-find stores for each node x, either a component node $c(x) \in X_i$ or a list of at most w_i component nodes, leading to at worst 2^{w_i} different lists per node, giving $2^{w_i^2}$ possible combinations of lists. Also there are 2^{w_i} possible binary vectors $\mathbf{v}$ of length w_i, therefore there are at worst $2^{w_i^2} 2^{w_i}$ elements in $\tilde{\mathcal{M}}(i)$ (this enumeration includes invalid matchings and incoherent pairs $(\mathbf{v}, \mathbf{uf})$).

The time complexity is dominated by the execution at the join bag where pairs of equivalences classes from $\tilde{\mathcal{M}}(j)$ and $\tilde{\mathcal{M}}(k)$ have to be combined. Therefore we must square the number of equivalence classes in each set: the complexity for a join bag is $O(4^{w^2+w} \cdot w^3 \cdot \log(w))$ (please refer to the next section for details). Since there are $O(n)$ bags in a nice tree decomposition, the total execution time is in $O(4^{w^2+w} \cdot w^3 \cdot \log(w) \cdot n)$. Finally, as already stated in §4(a), for bounded treewidth computing a tree decomposition and a nice tree decomposition is linear. Therefore the whole process from the tree decomposition to the resulting maximum alternating cycle-free matching is fixed-parameter tractable in the treewidth. Note that neither the decomposition nor the algorithm use the fact that the graph is bipartite.

**(d) Algorithm for ALTERNATING CYCLE-FREE MATCHING: step by step**

The algorithm visits the bags of the nice tree decomposition bottom-up from the leaves to the root evaluating the corresponding mappings in each step according to the following rules (Figure 8).

Leaf bag

The set of matchings $\tilde{\mathcal{M}}(i)$ of a leaf bag $X_i = \{x\}$ is trivial with a unique empty matching $\tilde{M}$ represented by $\mathbf{v}(\tilde{M}) = [0]$, and $\mathbf{uf}(\tilde{M})(x)$ defined as an empty list, associated with $m(\tilde{M}) = 0$.

Introduce bag

Let $X_i = X_j \cup \{x\}$ be an introduce bag with child bag X_j. The set of valid matchings $\tilde{\mathcal{M}}(i)$ is built from $\tilde{\mathcal{M}}(j)$ by introducing x in each matching $\tilde{M} \in \tilde{\mathcal{M}}(j)$, generating several possible matchings $\tilde{M}'$. We can always introduce x as an unmatched node, then $\tilde{M}$ is extended on x by setting $\mathbf{v}(\tilde{M}')_{|x} = 0$ and updating $\mathbf{uf}(\tilde{M}')$ with the ordered list of components for each matched neighbor of x. In addition, for each unmatched neighbor $y \in X_j$, we can introduce x as a matched node in the following way. We match both x and y in $\tilde{M}$ and set $\mathbf{v}(\tilde{M}')_{|x} = 1$ and $\mathbf{v}(\tilde{M}')_{|y} = 1$. If the intersection of the list of neighbor components of x and y is empty, then the matching of x and y does not create cycle. In this case $\tilde{M}'$ is a valid extension of $\tilde{M}$. The update of the union-find structure must then reflect the extensions of all alternating paths through arc $\{x, y\}$. We perform in $\mathbf{uf}(\tilde{M}')$ a union operation for x and all its matched neighbors (including y), and for y and all its matched neighbors. We also add the merged component index $c(x)$ to the list of neighbor components of each unmatched neighbor of x and y. Then we include all valid extensions $\tilde{M}'$ to $\tilde{\mathcal{M}}(i)$, reducing $\mathbf{v}(\tilde{M}')$ by calling find for each node and neighbor component list entry, and we set $m(\tilde{M}) = m(\tilde{M}')$ for all extensions $\tilde{M}'$ of $\tilde{M}$.
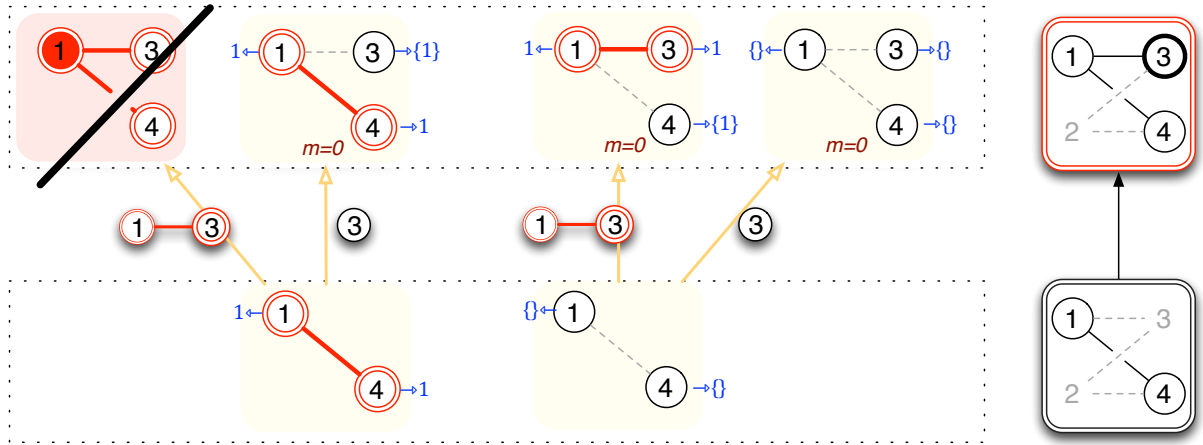
Figure 5: *Detail of the decisions at an introduce bag (the nomenclature is illustrated in Figure 8).*

Running time There are at most $2^{w_i^2 + w_i}$ extended matchings $\tilde{M}'$ for bag X_i (including all invalid ones), where $w_i = |X_i| = |X_j| + 1$ (a new possible matching can be generated only once). Each new matching is validated by a direct lookup at $\mathbf{v}(\tilde{M}')$ and ordered list comparison, leading to a linear time $O(w_i)$. The update of each structure requires constant time for each matched neighbor of x and almost linear time $O(w_i)$ plus the sorted insertion $O(w_i \cdot \log(w_i))$ for each unmatched neighbor, and there are at most w_i neighbors in the bag. Thus, the total running time of an introduce bag is in $O(2^{w_i^2 + w_i} \cdot w_i^2 \cdot \log(w_i))$.

Forget bag

Let $X_i = X_j \setminus \{x\}$ be a forget bag with child bag $X_j \ni x$. While the set of all possible matchings on $X_i \cup F_i$ does not change ($\mathcal{M}(j) = \mathcal{M}(i)$), the equivalence relation $\sim_i$ possibly identifies more matchings than $\sim_j$. For each matching $\tilde{M} \in \tilde{\mathcal{M}}(j)$, a new matching M' is obtained by deleting coordinate x of $\mathbf{v}(\tilde{M})$. If $c(x) = x$, $\mathbf{uf}(\tilde{M})$ needs to be updated. To do so, the set of nodes X_i is traversed twice, once to look for node $y \neq x$ of minimal index such that $c(y) = c(x)$ (possibly, y does not exist), and a second time to replace x by y each time x is used as a component index. If x was unmatched in $\tilde{M}$ (i.e., $\mathbf{v}(\tilde{M})_{|x} = 0$), then we set $m(M') = m(\tilde{M}) + 1$, otherwise we set $m(M') = m(\tilde{M})$. Once the set $\mathcal{M}(j)'$ of all the generated M' is computed, $\tilde{\mathcal{M}}(i)$ is obtained as the quotient of $\mathcal{M}(j)'$ by $\sim_i$, the equivalence relation on X_i. More precisely, each pair $(M', M'') \in \mathcal{M}(j)'^2$ is tested for equality on both $\mathbf{v}$ and $\mathbf{uf}$. If they are equal, one with the lowest m is defined to be the new representative in $\tilde{\mathcal{M}}(i)$.

Running time Each new matching M' is obtained from a single element of $\tilde{\mathcal{M}}(j)$ in worst-case time $O(w_i^2 \cdot \log(w_i))$. Equivalent matchings are detected on-the-fly when filling the hash structure of $\tilde{\mathcal{M}}(i)$, and each equivalence test is linear in w_i^2. The complexity is thus in $O(2^{w_j^2 + w_j} \cdot w_j^2 \cdot \log(w_j))$.
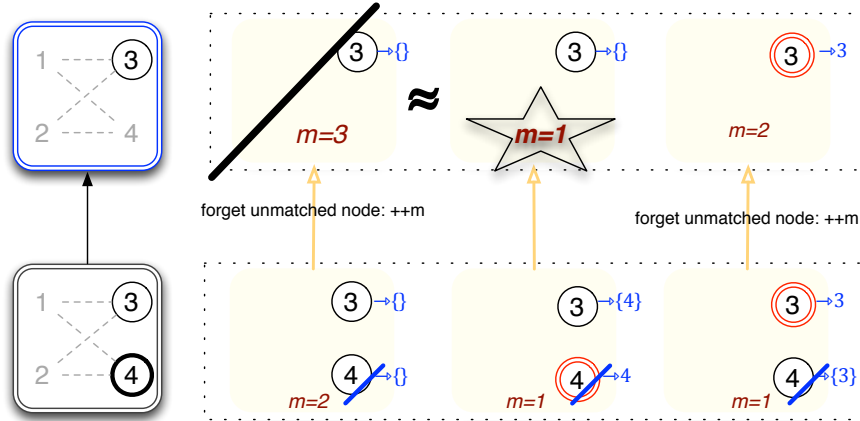

Figure 6: *Detail of the decisions at a forget bag.*

Join bag

Let $X_i = X_j = X_k$ be a join bag with child bags X_j and X_k. The matchings of $\mathcal{M}(i)$ are generated by combining all the pairs of matchings $(M, M') \in \mathcal{M}(j) \times \mathcal{M}(k)$. A combination is valid if and only if it satisfies both the matching and cycle-free conditions. The matching condition says that a node cannot be matched in both M and M', which is checked by a logical AND operation ($\mathbf{v}(M) \ AND \ \mathbf{v}(M')$). The cycle-free condition is checked with the union-find structures M and M': the combination is valid if no node of the component of a matched node in $\mathbf{uf}(M)$ is neighbor of the same component in $\mathbf{uf}(M')$ and vice versa, each test requiring $O(w_i^2)$ per component. If a combination is valid, its structure M'' is defined by $\mathbf{v}(M'') = \mathbf{v}(M) \ OR \ \mathbf{v}(M')$. The union-find structure is initialized from $\mathbf{uf}(M)$, and updated as the introduce bag for each matched node of M'. Finally, $m(M'') = m(M) + m(M')$. As in the forget bag, two combinations may result in equivalent matchings, and we must compare them pairwise and choose the representative with the lowest number of unmatched forgotten bags. Note that the sets of forgotten nodes of X_j and forgotten nodes of X_k have to be disjoint by the *coherence* of Definition 3 and hence no forgotten node can be counted twice in this setting. Furthermore, all possible combinations of matched and unmatched nodes are enumerated in $\mathcal{M}(j)$ and $\mathcal{M}(k)$ and hence no possible matching is overlooked.
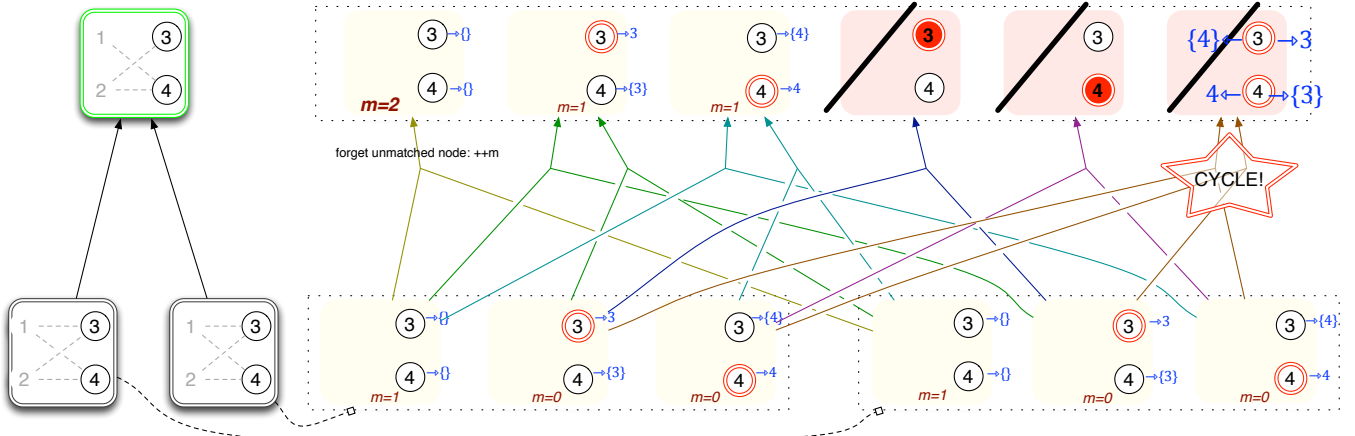
Figure 7: *Detail of the decisions at a join bag.*

Running time Each pair of matchings is validated and updated in time $O(w_i \cdot w_i^2 \cdot \log(w_i))$. The comparison and the choice of representative is done on-the-fly when filling the hash structure of $\mathcal{M}(i)$. There are at worst $(2^{w_i^2 + w_i})^2$ pairs. Thus, the complexity of the join bag dominates all other running times. Therefore, the complexity of the algorithm is in $O(4^{w_i^2 + w_i} \cdot w_i^3 \cdot \log(w_i))$ per bag.
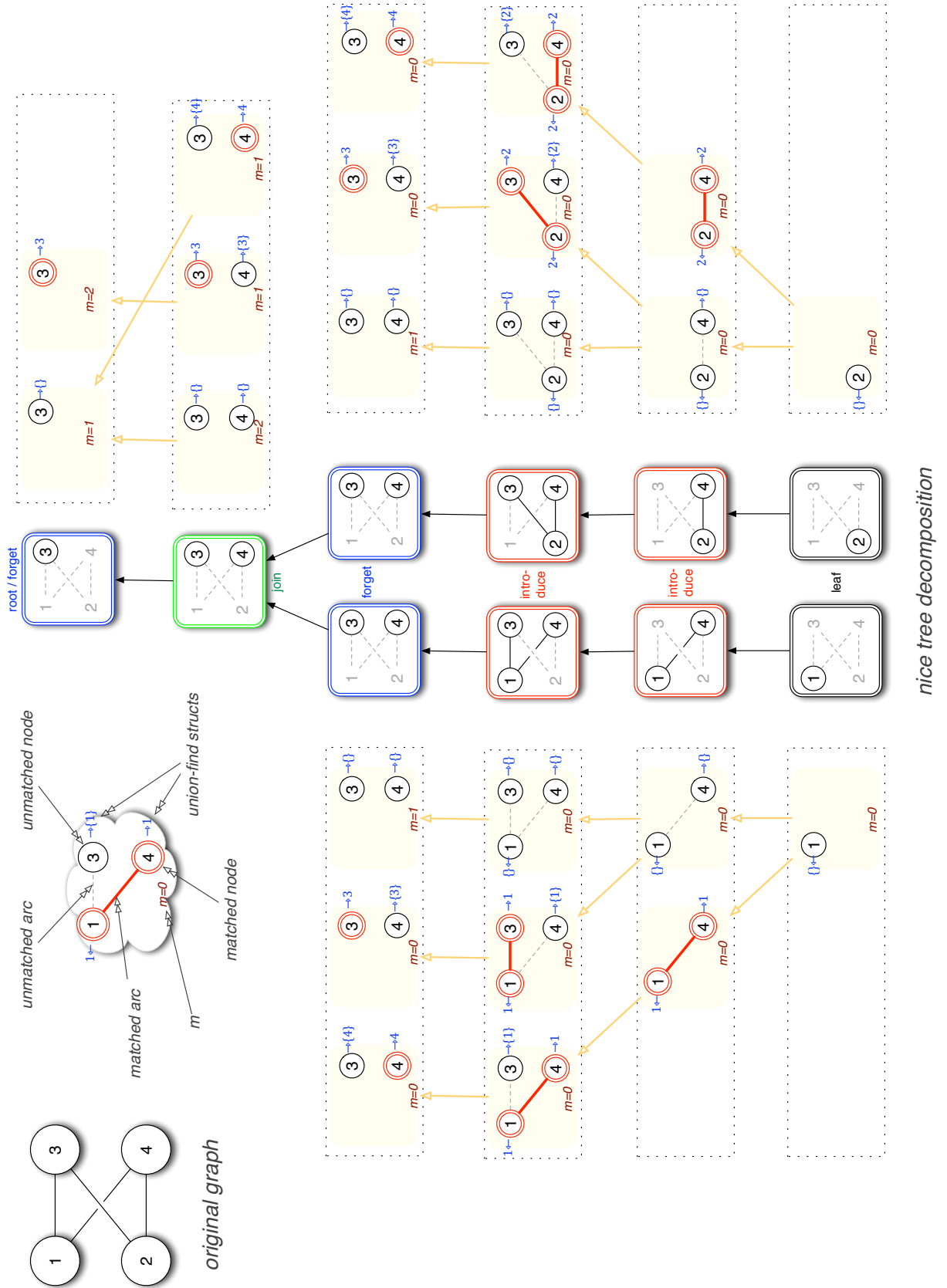
Figure 8: *Algorithm execution on a small bipartite cycle (top left) with its nice tree decomposition (center). At each bag, a set of matchings $\tilde{\mathcal{M}}(i)$ is generated according to the bag type. $\tilde{\mathcal{M}}(i)$ is represented on the side of each bag, with the nomenclature illustrated at the top of the figure.*

Root bag

Let $X_r = \{x\}$ be the root of T. $\mathcal{M}(r)$ consists of at most two matchings $\mathbf{v}(\tilde{M}) = [0]$ or $\mathbf{v}(\tilde{M}') = [1]$, where $\mathbf{uf}(\tilde{M})$ is an empty list and $\mathbf{uf}(\tilde{M}')$ defined by $c(x) = x$. It follows that the minimum number of unmatched nodes for any alternating cycle-free matching of G is given by $m = \min\{m(\tilde{M}) + 1, m(\tilde{M}')\}$, and the maximum size of an alternating cycle-free matching is given by $(n - m)/2$ where $n = |N|$ denotes the number of nodes of G.

Total Running Time   The total time complexity of the algorithm is bounded above by the running time of the join bag. Since there is a linear number of bags, and since for every bag X_i we have $|X_i| \leq tw(G) + 1 = w + 1$, the total time complexity of the algorithm described above is

$$O(4^{w^2 + w} \cdot w^3 \cdot \log(w) \cdot n).$$

(e) Correctness of the Algorithm

We must check that the algorithm, without the grouping, considers every possible alternating cycle-free matching in G and that the grouping occurring at the forget and join bags does not discard the maximum matching.

The *node coverage* and *arc coverage* properties of nice tree decompositions (Definition 3) ensure that each node is processed and each arc is considered for inclusion in the matching at one introduce node. Since the introduce node discards only matchings that violate either the matching or the cycle condition, and these violations cannot be legalized by further extensions or combinations of the matchings, all possible valid matchings are considered.

Now, consider two matchings M and M' that are grouped together and represented by $\tilde{M}$ at a forget or join bag X_i. Further along in the algorithm, the representative $\tilde{M}$ is then extended or combined with other matchings to form new valid matchings $\tilde{M}'$. The *coherence* property of Definition 3 assures that no neighbor of a newly introduced node can be a forgotten node, so the extension or combination only modifies matchings M and $\tilde{M}$ on nodes of X_i, which are represented in the structure of $\tilde{M}'$. Hence, the valid matchings $\tilde{M}'$ actually represent all the valid extensions and combinations of M and $\tilde{M}$. The grouping thus generates all valid and relevant representatives of matchings in order to find a maximum alternating cycle-free matching. Moreover, in case M and M' are equivalent and both with the lowest number of forgotten unmatched nodes, choosing M or M' as representative leads to the exact same extensions and combinations.

Finally, let M_m be the alternating cycle-free matching of maximum size of G. In each bag the corresponding matching must be one of the matchings with the lowest number of unmatched nodes within its equivalence class $\tilde{M}_m \in \tilde{\mathcal{M}}(i)$. Otherwise, a matching in the same class $\tilde{M}_m$, extended and combined as M_m in the sequel of the algorithm would give rise to a matching with fewer unmatched nodes. Therefore, the choice of the representative at the forget and join bags never discards a future alternating cycle-free matching of maximum size.

(f) Treewidth of the dual graph

Up to this point, we have been dealing primarily with simplicial complexes and their spines. We now turn our attention to simplicial triangulations of 3-manifolds and a more natural parameter associated to them.

Definition 5 (Dual graph). *The dual graph of a simplicial triangulation of a 3-manifold $\mathcal{T}$, denoted $\Gamma(\mathcal{T})$, is the graph whose nodes represent tetrahedra of $\mathcal{T}$, and whose arcs represent pairs of tetrahedron faces that are joined together.*

We show that, if the treewidth of the dual graph is bounded, so is the treewidth of the spine, as stated by the following theorem.

Theorem 7. *Let G be the spine of a simplicial triangulation of a 3-manifold $\mathcal{T}$. If $\mathbf{tw}(\Gamma(\mathcal{T})) \leq k$, then $\mathbf{tw}(G) \leq 10k + 9$.*

Proof. Let T be a tree decomposition of the dual graph, where each bag X_i contains less than or equal $k + 1$ tetrahedra. We show how to construct a tree decomposition T' of the spine of $\mathcal{T}$, modeled on the same underlying tree as T, in which each bag X_i' contains less than or equal $10(k + 1)$ edges and triangles.

For each bag X_i of T, we simply define the bag X_i' to contain all edges and triangles of all tetrahedra in X_i. It remains to verify the three properties of a tree decomposition (Definition 3).

Node coverage   It is clear that every edge or triangle in the spine belongs to some bag X_i', since every edge or triangle is contained in some tetrahedron δ, which belongs to some bag X_i.

Arc coverage   Consider some arc in the spine. This must join a triangle t to an edge e that contains it. Let δ be some tetrahedron containing t; then δ contains both t and e, and so if X_i is a bag containing δ then the corresponding bag X_i' contains the chosen arc in the spine (joining t with e).

n	# triangulations		spine				dual graphs			
			min	max	average		min	max	average	
1	4	(3)	1	2	1.50	(1.67)	0	0	0.00	
2	17	(12)	1	3	2.47	(2.42)	1	1	1.00	
3	81	(63)	1	3	2.51	(2.49)	1	2	1.60	(1.52)
4	577	(433)	1	5(4)	2.77	(2.73)	1	3	1.91	(1.87)
5	5184	(3961)	1	6(5)	2.95	(2.95)	1	4	2.16	(2.18)
6	57753	(43584)	1	6	3.16	(3.19)	1	4	2.31	(2.35)
7	722765	(538409)	1	7	3.35	(3.40)	1	4	2.45	(2.50)

Table 1: *Treewidths of the spine (left) and of the dual graphs (right) of closed generalized triangulations up to 7 tetrahedra. The values in brackets are for 1-vertex triangulations.*

Coherence   Here we treat edges and triangles separately.

Let t be some triangle in the simplicial complex. We must show that the bags containing t correspond to a connected subgraph of the underlying tree. If t is a boundary triangle, then t belongs to a unique tetrahedron δ, and the bags X_i' that contain t correspond precisely to the bags X_i that contain δ. Since the tree decomposition T satisfies the connectivity property, these bags correspond to a connected subgraph of the underlying tree. If t is an internal triangle, then t belongs to two tetrahedra δ_1 and δ_2, and the bags X_i' that contain t correspond to the bags X_i that contain *either* δ_1 or δ_2. By the connectivity property of the original tree, the bags containing δ_1 describe a connected subgraph of the tree, and so do the bags containing δ_2. Furthermore, there is an arc in the dual graph from δ_1 to δ_2, and therefore some bag X_i contains *both* δ_1 and δ_2. Thus the union of these two connected subgraphs is another connected subgraph, and we have established the connectivity property for t.

Now let e be some edge of the simplicial complex. Again, we must show that the bags containing e correspond to a connected subgraph of the underlying tree. This is simply an extension of the previous argument. Suppose that e belongs to the tetrahedra $\delta_1, \ldots, \delta_m$ (ordered cyclically around e). Then for each δ_j, the bags X_i that contain δ_j describe a connected subgraph of the underlying tree, and the bags X_j' containing e describe the union of these subgraphs, which we need to show is again connected. This follows because there is a sequence of arcs in the dual graph (δ_1, δ_2), (δ_2, δ_3) and so on; from the tree decomposition T it follows that the subgraph for δ_1 is connected, the subgraph for δ_2 is connected, the subgraph for δ_3, and so on. Therefore the union of these subgraphs is itself connected.   □

5   Empirical Observations

In §3 we have seen that the problem of finding optimal Morse matchings is hard to solve in general. On the other hand, in §4, we proved that in the case of a small treewidth of the spine of a 2-dimensional complex or, equivalently, in the case of a bounded treewidth of the dual graph of a simplicial triangulation of a 3-manifold, finding an optimal Morse matching becomes easier. Up to a certain scaling factor, the results stated in §4 hold for generalized triangulations as well (also, note that the notion of a spine or the dual graph can be extended in a straightforward way to generalized triangulations).

Given this situation, natural questions to ask are the following: (i) Which 3-manifolds can be triangulated with a dual graph with small treewidth, i.e., which 3-manifolds admit triangulations where our fixed parameter tractable algorithm can be practical? What is a *typical* value for the treewidth of the respective graphs of (ii) small generic generalized triangulations of 3-manifolds, and (iii) small generic simplicial triangulations of 3-manifolds?

To answer (i) we will focus on generalized triangulations and note, that any such triangulation can be transformed into a simplicial triangulation by a 2-fold barycentric subdivision, keeping the treewidth of the dual graph within a constant of the treewidth of the dual graph of the original triangulation.

It is a well-known fact in 3-manifold topology that any Dehn filling along a 2-triangle torus boundary can be triangulated by so-called *layered solid tori* [9] which have dual graphs of tree-width one. This means that any core manifold with torus boundary components, e.g., any knot- or link-complement, can be closed by adding arbitrary layered solid tori. In particular, each such core manifold translates into a class of infinitely many distinct 3-manifolds with triangulations of unbounded size all of which have dual graphs of uniformly bounded treewidth.

For instance, all members of the large class of *Seifert fibred spaces* [13] – which contains many standard geometric 3-manifolds – admit triangulations consisting of a small region containing the base orbifold together with families of exceptional fibres realised by layered solid tori. In particular, in the relatively large subclass of Seifert fibred spaces with base orbifold S^2, these standard triangulations all have dual graphs of treewidth two.

But also non-geometric classes of 3-manifolds can be identified to have triangulations with dual graphs of very small treewidth. As an example, for the very general class of so-called *graph manifolds* [26, 50], there exist triangulations which

		spine			dual graphs		
n	# triangulations	min	max	average	min	max	average
5	1	6	6	6.00	4	4	4.00
6	2	$\leq$ 7	$\leq$ 8	$\leq$ 7.50	4	5	4.50
7	5	$\leq$ 8	$\leq$ 11	$\leq$ 9.40	4	6	5.00
8	39	$\leq$ 8	$\leq$ 14	$\leq$ 11.23	4	7	5.74
9	1297	$\leq$ 8	$\leq$ 18	$\leq$ 13.55	4	9	7.01
10	249015	$\leq$ 8	$\leq$ 22	$\leq$ 16.33	$\leq$ 4	$\leq$ 13	$\leq$ 8.99

Table 2: *Upper bounds and exact values for the treewidths of the spine (left) and of the dual graph (right) of simplicial triangulations of 3-manifolds up to 10 vertices.*

essentially have dual graphs of treewidth within a constant factor of the treewidth of the underlying graph. Note that the complexity of graph manifolds is rapidly increasing as the graph grows larger since any node of the graph corresponds to an arbitrary Seifert fibred space. In addition, following recent results of [8] there are reasons to believe that even larger infinite classes of topological 3-manifolds admit triangulations whose treewidths are below provable upper bounds. Investigating these upper bounds is work in progress.

In order to answer (ii) and (iii) we conducted a series of computer experiments computing the treewidth of the relevant graphs (i.e., the spine and the dual graph) of all closed generalized triangulations of 3-manifolds up to 7 tetrahedra [28], and all simplicial triangulations of 3-manifolds up to 10 vertices [11]. The computer experiments were done using LibTW [5] to compute the treewidth / upper bounds for the treewidth, with the help of the GAP package simpcomp [57, 58] and the 3-manifold software Regina [27, 55]. We report the minimal, maximal and average treewidths of all triangulations with the same number of tetrahedra in Table 1 and of all simplicial triangulations with the same number of vertices in Table 2. Furthermore, in Table 3 we list the tree-width of the spines of a number of small 2-dimensional simplicial complexes with various properties identified by Hachimori, and Takeuchi [39, 53].

Regarding the treewidth of generalized triangulations of 3-manifolds, we observe that there is a large difference between the average treewidth and the maximal treewidth for both the dual graph and the spine. In particular, the average treewidth appears to be relatively small. Moreover, there is only a slight difference between the data for general closed triangulations and 1-vertex triangulations. This fact is somehow in accordance with our intuition since the number of 0-dimensional simplices should neither directly affect the spine nor the dual graph of a generalized triangulation.

On the other hand, the gap between the maximum treewidth and the average treewidth in the case of simplicial triangulations of 3-manifolds is relatively small compared to the data for generalized triangulations. At this point it is important to note that, while the data concerning the spines for simplicial complexes only consists of upper bounds, experiments applying the algorithm for the upper bound to smaller graphs and then computing their real treewidths suggest that these upper bounds (on average) are reasonably close to the exact treewidth.

Further analysis shows that the average treewidth of the spines for both generalized and simplicial triangulations of 3-manifolds is mostly less than twice the treewidth of the dual graph, and hence much below the theoretical upper bound given by Theorem 7. Also, the ratio between these two numbers appears to be more or less stable for all values shown in Tables 1 and 2. This can be seen as experimental evidence that for triangulated 3-manifolds the treewidth of the dual graph is responsible for the inherent difficulty to solve ERASABILITY and related problems.

Despite the small values of n in our tables, the observations made above regarding Question (i) lead us to believe that the patterns of small treewidth will continue for larger n and thus add relevance to the fixed parameter tractability results presented in this article and the treewidth as a useful parameter for topological problems.

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complex	shell.	ext. s.	constr.	vtx. dec.	f-vector	w
Dunce hat, [19]			No		$(8, 24, 17)$	≤ 6
[24, Exerc. 7.37]		No			$(6, 15, 11)$	5
[39]	Yes	No			$(7, 19, 13)$	≤ 5
[39]	No		Yes		$(12, 37, 26)$	≤ 5
[39]	No		Yes		$(13, 39, 27)$	≤ 5
[39]	No		Yes		$(10, 31, 22)$	≤ 5
[16]		No			$(7, 20, 14)$	≤ 4
[16]	Yes			No	$(6, 15, 10)$	4
[53]	Yes	No			$(6, 14, 9)$	3
[53]	Yes	No			$(6, 14, 9)$	3
[53]	Yes	No			$(6, 15, 10)$	4
[53]	Yes	No			$(6, 15, 10)$	4
[53]	Yes	No			$(6, 15, 10)$	4
[53]	Yes	No			$(6, 15, 10)$	4
[53]	Yes	No			$(6, 15, 10)$	4
[53]	Yes	No			$(6, 15, 11)$	5
[53]	Yes	No			$(6, 15, 11)$	5
[53]	Yes	No			$(7, 17, 11)$	4
[53]	Yes	No			$(7, 18, 12)$	≤ 4
[53]	Yes			No	$(6, 15, 10)$	4
[53]	Yes			No	$(6, 15, 10)$	3

Table 3: *Treewidths (w) of the spine of some 2-dimensional simplicial complexes of particular interest for discrete Morse theory, together with information on their shellability (sh.), extended shellability (ext. s.), vertex decomposability (vtx. dec.), and f-vector (the i-th entry of the f-vector of a simplicial complex denotes the number of i-dimensional simplices in the complex).*

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