Instituto de Matemática Pura e Aplicada

Doctoral Thesis

FOLIATING MARCHING CUBE'S CASES IN DIMENSIONS THREE AND FOUR

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To Fernanda, my beloved daughter.

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There is nothing more practical than a good theory.
Kurt Lewin

Abstract

Marching Cubes is the reference method when constructing the isosurface of a scalar field f sampled over a grid is the task at hand. After the original publication, numerous modifications and extensions of the method have been published. The majority of these modifications focuses mainly on variations to the shape of the cells of the grid, the dimension, or even, the number of categories used to classify the vertices of the grid (e.g., positive, negative, and zero). In this thesis, we address the problem in dimensions three and four. In dimension three, one of our main goals is to construct a much more complete case description based on linear interpolation, and consequently a better algorithm. To achieve this, we need to solve a primary problem: the classification of all the solutions of the equation $f(x, y, z) = 0$, where f is trilinear function. Once all the solutions are obtained and visualized, we are able to provide all the intersections of these solutions with a cube. Such intersections allow us to know all the possible formats (or portions of surface) that the solutions above describe within the cube. The correct list of all configurations with its respective surface naturally leads to an improvement of the correctness of the surface generation. Then, from the classification obtained in dimension three and the extension of the main ideas and results, we achieve to explore some aspects of the problem in dimension four. In this regard, two main contributions are presented. The first is a coarse classification of all the solutions of the equation $f(x, y, z, t) = 0$, where f is a quadri-linear function. The second consists of a primary marching hypercubes algorithm restrited to configurations not presenting null vertices nor internal ambiguities.

Keywords: Marching Cubes, Isosurface Extractors, 4D Visualization, Foliations, Interpolation, Discrete Topology.

Resumo

Marching Cubes é a metodologia de referência quando o problema requer a visualização da superfície de nível de um campo escalar amostrado nos vértices de um reticulado. Depois da publicação do algoritmo original, muitas modificações e extenções têm sido reportadas na literatura. A grande maioria delas tem estado focada em variações na dimensão e forma das células do reticulado, ou inclusive, no número de categorias usadas para classificar seus vétices, por exemplo, positivo, negativo e nulo. Neste trabalho, se estuda o problema em dimensões três e quatro. Em dimensão três, nosso principal objetivo é descrever os casos possíveis, de modo muito mais detalhado e específico, quando interpolação linear é usada, e como consequência obter um algoritmo muito mais robusto e eficiente. Para tal, nós temos que resolver um problema primário: a classificação de todas as soluções da equação $f(x, y, z) = 0$, onde f é uma função trilinear. De posse de tais soluções, nós podemos descrever todas as possíveis porções de superfícies que podem ser desenhadas dentro de um cubo a partir de uma função trilinear. A lista de todas as configurações de sinais dados aos vértices do cubo junto com a sua respectiva porção de superfície separando os vértices com diferente sinal conduze a uma melhora substancial no processo de geração da isosuperfcie. Baseados nos resultados obtidos em dimensão três e estendendo as principais ideias da filosofia adotada, nós começamos o estudo em dimensão quatro. Duas importantes contribuições são exibidas. A primeira delas consiste de uma classificação inicial das soluções da equação $f(x, y, z, t) = 0$, onde f é uma função de interpolação linear . A segunda é a introdução de um algoritmo primário que não trata nem com as ambiguidades internas nem com os vértices nulos.

Palavras-chave: Marching Cubes, Visualização de Superfícies, Visualização 4D, Foliações, Interpolação, Topologia Discreta

Contents

Introduction

Level surface extractors and implicit surface tilers have facilitated access to many types of data including, but not limited to medical images, physical simulations, 3D reconstruction, geological and petroleum exploration. Among all techniques that have been reported in literature, Marching Cubes is the most referenced methodology. The purpose of the algorithms that are based on this philosophy is to produce a level (hyper)surface S of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$, namely the set of those points in $\mathbb{R}^n$ that satisfy the equation $f(x_1, x_2, \ldots, x_n) = c$, for certain real constant c. The original Marching Cubes algorithm was introduced by Lorensen and Cline in 1987, [1], for the case in that $f : \mathbb{R}^3 \to \mathbb{R}$. Roughly speaking, the algorithm works as following:

1. The input of the algorithm is a signed cuberille grid
$$G = \{(v_{ijk}, \mathrm{sg}(f(v_{ijk}))) \,:\, v_{ijk} = (x_i, y_j, z_k)\}$$
obtained by sampling f on the points v_{ijk}'s, $1 \leq i \leq p$, $1 \leq j \leq q$, $1 \leq k \leq r$. Here, sg denotes the sign function.

2. The algorithm processes one cube at a time and uses linear interpolation to determine whether S intersects an edge. In this way the algorithm yields an approximation of the portion of S contained in each cube of G and produces a global result. The trick here is to match each cube to a representative case via a lookup table which contains the solution for each possible configuration.

This approach relies heavily on the lookup table. Frequently, the lookup table splits into three tables. The first table stores all the possible configurations of the (hyper)cube, that is, all the manners we can assign a sign to each vertex of the (hyper)cube. The n-dimensional (hyper)cube has 2^n vertices, and each vertex has usually assigned one of the signs positive or negative, obtaining a total 2^{2^n} configurations. Thus, we have 16 configurations in dimension two, 256 in dimension three, and 65 536 in dimension four. Observe that the number of configurations increases exponentially with dimension. Sometimes, in addition to positive and negative signs the category "null" is used leading to 3^{2^n} configurations. Once all the configurations were computed, we need to investigate all possible solutions for each configuration. An important point to note here is that there are configurations which admit more than one portion of surface as possible solution. In dimension two this situation arises when the diagonally-opposed vertices of the square have the same sign, while consecutive vertices have different sign. This phenomenon is known as *ambiguaties* and it is perhaps the biggest challenge for this approach. In higher dimensions we find two kind of ambiguities: (a) face ambiguities, and (b) internal ambiguities. The face ambiguities appears in a lower-dimensional face of the cube, while internal ambiguities is associated to the presence of a "tunnel" that connects two vertices along an internal diagonal. The cubes that present ambiguities require some additional test before deciding its corresponding solution. The second table stores the additional topological tests to be performed in each ambiguous configuration, and maps the possible results of the test into all subconfigurations. In the third table we can find the tiling (i.e., triangulation) that corresponds to each (sub)configuration. Such a tiling depends on linear interpolation and the topological tests. Hereafter, we refer to these tables as *configuration table*, *test table*, and *tiling table*, respectively.

The solutions of linear interpolation and Marching Square

This section provides the main results about marching square. To obtain the possible tilings for each configuration of marching square, we need to know two things: (1) all the solutions of the linear interpolation equation $P(x, y) = 0$, where $P(x, y) = b_0 + b_1x + b_2y + b_3xy$, $b_i \in \mathbb{R}$, $0 \leq i \leq 3$; and all intersections of the solutions of the previous equation with the square $[0, 1]^2$.

From the classification of algebraic curves, we know that if C is a solution of the linear interpolation equation $P(x, y) = 0$, then C is:

1. a hyperbola, if $b_3 \neq 0$. The hyperbola is degenerated (i.e., two orthogonal straight lines) if and only if $b_3b_0 - b_1b_2 = 0$.

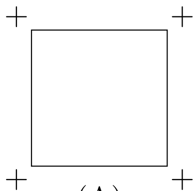
2. a straight line, if $b_3 = 0$ and at least one of the coefficients b_1 and b_2 is not zero.

3. a plane, if $b_0 = b_1 = b_2 = b_3 = 0$.

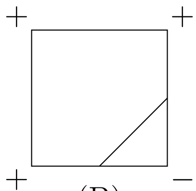
4. the empty set, if $b_0 \neq 0$ and all remaining coefficients are null.

It is an easy matter to check that this classification leads to the following representative tilings of the marching square:

Non-degenerated cases without zeros



(A)



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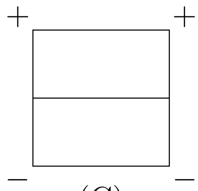

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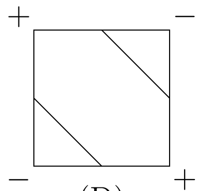

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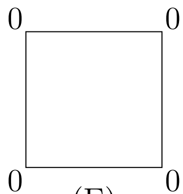
Non-degenerated cases with zeros



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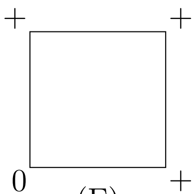

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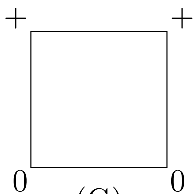
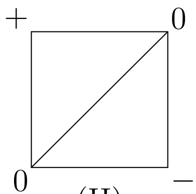
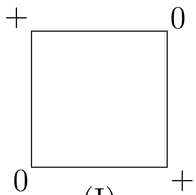


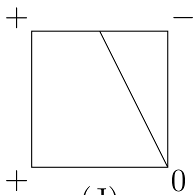
(G)



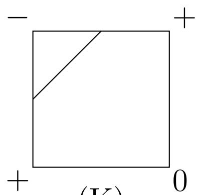
(H)



(I)

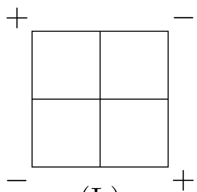


(J)



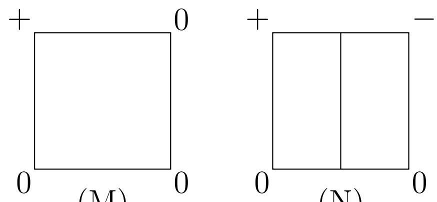
(K)

Degenerated cases without zeros



(L)

Degenerated cases with zeros



(M) (N)

Thus, the configuration table possesses fourteen cases (A)-(N), each of them grouping equivalent configurations. For example, case (A) stores exactly two configurations: (+,+,+,+) and (-,-,-,-), which are related by a reversing sign map. In turn, case (B) stores eight configuration, all of them equivalent to (+,-,+,+). One representative configuration of the remaining cases was shown above.

The test table stores the test (if any) we have to perform for the configuration of each case. The only cases that need to be tested are those presenting ambiguities.

The tiling table stores the tiling corresponding to each configuration.

In higher dimensions we proceed analogously. When constructing the configuration table, it is customary to gather those configurations that are the same up to a rigid movement of the space (e.g., rotations, reflections) or a map that reverses the sign of each vertex of the (hiper)cube. We can think of the configuration table as the collection of orbits of the action of the direct product of these groups. The construction of the configuration table is associated to the counting cases problem. The main task to be solved in this latest problem is to count all the non-equivalent configurations of signs that can be attributed to the vertices of the (hyper)cube. In other words, the counting problem investigates the number of orbits of the action of the afore-mentioned group over the set of all the configurations of signs that can be assigned to the vertices of the (hyper)cube. Counting cases approach is also concerned with other statements of the marching cubes problem which include variations in the dimension, the shape of the cells in the grid, the number of "signs," etc.. In this regard, the article of Banks and Linton, [11], provides valuable information.

On the other hand, the tiling we associated to each configuration of signs of the (hiper)cube corresponds with the intersection of a solution of the interpolation equation $P(x_1, x_2, \ldots, x_n) = 0$ with the (hiper)cube $[0,1]^n$. This leads to a much more difficult task — the classification of all the solutions of the linear interpolation equation. In the bi-dimensional case, the classification of the solutions of the equation $P(x, y) = 0$ is easily deduced from the classification of algebraic curves such as we showed above. In dimension three, a classification of the solutions of the linear interpolation equation $P(x, y, z) = 0$ was introduced by Nielson [2]. In higher dimensions, this is an open problem.

Contributions and organization

Nowadays there are countless variants of the original algorithm in dimensions two, three and four, although undoubtedly the majority of them focus on the three-dimensional case. Aside from the classification, perhaps the greatest contribution in dimension two is the asymptotic decider [4], the simplest test to resolve the ambiguous cases. In dimension three we have already mentioned two of the main works: the original article of marching cubes, and the classification introduced by Nielson [2]. Another important input is the work of Lewiner et al. [3] in which a full implementation of the Nielson's lookup table is introduced. Other articles have been published extending the method to other data input, and also to improve its computational efficiency. Among the most important works in the first of these task we can cite the article of Weber et al. [5] which expands on the marching cubes principle to rectilinear grid data with multi-resolution regions. The extension of the method to non-rectangular data has also been carried out in several ways. For instance, extensions of the marching cubes method for unstructured grids have been proposed [6], and the approach of tetrahedral cells has been addressed by several authors [7, 8, 9, 10]. Among the approaches devoted to accelerate the algorithms we can find those with interval-based representations [12, 13, 14], those based on hierarchies [15, 16, 17], those based on propagation [18, 19, 20], those based on parallel and distributed processing [21, 22]; just to mention some examples. The reader interested in more details can consult the survey of this subject presented by Newman and Yi [23]. In dimension four, Bhaniramka et al. introduced an algorithm which automatically produces a tiling of each hypercube by using a convex-hull algorithm [24]. A different approach was presented by Robert and Hill [25].

In this work, we address the problem in dimensions three and four. The main contributions are:

1. We face the classification of the solutions of the linear interpolation equation by looking to some suitable foliations. As we already mentioned, one of the first steps towards the construction of marching cubes algorithm is the classification of all the possible solutions of the interpolation equation. Our approach via foliations allows us to easily generalize the thecniques to higher dimensions. In dimension three, we

get a similar result to that presented by Nielson; however, the foliations brings up some interesting elements. These topics are presented in Chapter 1.

2. One of the advantages of our approach is that it is also very useful when computing the intersections of the solutions of the linear interpolation with the (hiper)cube. This is evidenced in Chapeter 2, where a complete list of such intersections, including those with zeros and those which are denerated, is presented. As a consequence, a complete look up table for the marching cubes algorithm is given.

3. An improved marching cubes algorithm is presented in Chapter 3.

4. The approach of foliations is extended to dimension four in order to obtain the classification of the solutions of the linear interpolation equation $P(x, y, z, t) = 0$. The details are treated in Chapter 4.

5. Finally, Chapter 5 introduces a primary Marching Hypercubes algorithm. Because we do not have yet the entire classification of the solutions of the linear interpolation equation, the tiling table of this algorithm is incomplete.

Thus, in addition to the introduction, the thesis is organized into six chapters. The first is devoted to the classification of all solutions of the linear interpolation equation in dimension three. Chapter 2 uses the results obtained in the previous to construct a complete lookup table for the algorithm that is presented in Chapter 3. Following a similar reasoning as in the three dimensional case, Chapter 4 presents a coarse classification of the solutions of the linear interpolation equation in dimension four. Chapter 5 is intended to the development of a primary Marching Hypercubes algorithm. Finally, we present some concluding remarks and future works.

CHAPTER 1

Linear Interpolation in $\mathbb{R}^3$

The main goal of this chapter is to provide a complete classification of the solutions of the linear interpolation equation:

$$P(x,y,z) = b_0 + b_1x + b_2y + b_3z + b_4xy + b_5xz + b_6yz + b_7xyz = 0, \quad (1.1)$$

where all the coefficients are real numbers. Such classification consists in partitioning the set of these solutions in minor disjoint subsets so that two solutions lie together in some of these subsets if and only if they are equivalent in some sense. We are interested in that equivalent solutions, as well as sharing the same topological properties, be visually indistinguishable. To accomplish this, we define the equivalence by means of maps which only involve translations, changes in the scale, reflections, and rotations. Thus, we say that the solutions $S = \{(x,y,z) \in \mathbb{R}^3 : P(x,y,z) = 0\}$ and $S' = \{(x,y,z) \in \mathbb{R}^3 : P'(x,y,z) = 0\}$ are equivalent if and only if there exists an affine isomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ which is the composition of a diagonal isomorphism $\varphi : (x,y,z) \mapsto (ax+d, by+e, cz+f)$ $(a,b,c \neq 0)$ with a permutation σ of the set $\{x,y,z\}$ such that $P' = C(P \circ \Phi)$, for certain real constant C. Φ is often called *equivalence* between S and S'.

Here are some relationships between the coefficients of P and the coefficients of $P \circ \varphi$ and $P \circ \sigma$, respectively. If $P \circ \varphi(x,y,z) = B_0 + B_1x + B_2y + B_3z + B_4xy + B_5xz + B_6yz + B_7xy$, then

$$\begin{aligned}
B_0 &= b_0 + b_1 d + b_2 e + b_3 f + b_4 de + b_5 df + b_6 ef + b_7 def. && (1.2)\\
B_1 &= a(b_1 + b_4 e + b_5 f + b_7 ef). && (1.3)\\
B_2 &= b(b_2 + b_4 d + b_6 f + b_7 df). && (1.4)\\
B_3 &= c(b_3 + b_5 d + b_6 e + b_7 de). && (1.5)\\
B_4 &= ab(b_4 + b_7 f). && (1.6)\\
B_5 &= ac(b_5 + b_7 e). && (1.7)\\
B_6 &= bc(b_6 + b_7 d). && (1.8)\\
B_7 &= b_7 abc && (1.9)
\end{aligned}$$

Now, if we compose P with a permutation σ, then the sequence of coefficients of $P \circ \sigma$ is a permutation of the sequence of coefficients of P. The table below shows the cases of the permutations σ_{xy}, σ_{xz}, and σ_{yz}.

Table 1.1: Coefficient permutations.

P	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7
$P \circ \sigma_{xy}$	b_0	b_2	b_1	b_3	b_4	b_6	b_5	b_7
$P \circ \sigma_{xz}$	b_0	b_3	b_2	b_1	b_6	b_5	b_4	b_7
$P \circ \sigma_{yz}$	b_0	b_1	b_3	b_2	b_5	b_4	b_6	b_7

1.1 Invariants and foliations

The classification is based on few invariants. An invariant is a property which is preserved under any equivalence Φ, thereby a solution meets this property if and only if so does any other solution equivalent to it. The primary invariant is the set of all intersections of S with the planes normal to a principal axis θ. These intersections determine a foliation $F_\theta := \{S \bigcap \{\theta = \theta_0\}, \theta_0 \in \mathbb{R}\}$ of S which provides some valuable geometric information about S. Here and subsequently, the term *foliation* means a decomposition of a surface S as union of all its intersections with planes perpendicular to a given axis. Each one of these intersections is called *leave* of the foliation.

The following result describes all the possibilities we have for the leaves of F_x.

Lemma 1. *For each leave $S \bigcap \{x = x_0\}$ of the foliation F_x, we have one the following possibilities:*

1. *$S \bigcap \{x = x_0\}$ is a hyperbola. This occurs when $b_7x_0 + b_6 \neq 0$. The hyperbola will be degenerated (i.e., two orthogonal straight lines) if and only if x_0 satisfies the equation $(b_1b_7 - b_4b_5)x^2 + (b_0b_7 + b_1b_6 - b_3b_4 + b_2b_5)x + (b_0b_6 - b_2b_3) = 0$.*

2. *$S \bigcap \{x = x_0\}$ is a straight line. This situation appears if $b_7x_0 + b_6 = 0$ and at least one of the numbers $b_5x_0 + b_3$ and $b_4x_0 + b_2$ is not zero.*

3. *$S \bigcap \{x = x_0\}$ is a plane. This occurs if $b_7x_0+b_6 = b_5x_0+b_3 = b_4x_0+b_2 = b_1x_0 + b_0 = 0$.*

4. *$S \bigcap \{x = x_0\}$ is the empty set, which happens when $b_7x_0 + b_6 = b_5x_0 + b_3 = b_4x_0 + b_2 = 0$ and $b_1x_0 + b_0 \neq 0$.*

Proof. Writing P in the form $P(x, y, z) = (b_1x + b_0) + (b_4x + b_2)y + (b_5x + b_3)z + (b_7x + b_6)yz$, the result follows by using the classification of the bi-dimensional case. □

Lemma 1 has a similar formulation when a plane ortogonal either to the axis y or to the axis z is considered. To obtain the conditions over the coefficients, it suffices to apply the correspondence among the coefficients exhibited in Table 3.

The next result gives us information about the distribution of the leaves of F_θ.

Lemma 2. *All the leaves of F_θ, except at most for one, have the same classification as a solution of the bi-dimensional case.*

Proof. We can assume without loss of generality that $\theta = x$. Using the expression $P(x, y, z) = (b_1x+b_0)+(b_4x+b_2)y+(b_5x+b_3)z+(b_7x+b_6)yz$, we can note that the coefficients depend linearly on the parameter x and hence those non-identically null can vanish at most once. So, using Lemma 1 we can conclude that if at least one of the coefficient b_6 and b_7 is non-null, then all the leaves of F_x are hyperbolas when $b_7x + b_6$ does not vanish for all value of x; otherwise all the leaves of F_x, except for one, are hyperbolas. The same reasoning applies for the remaining cases. □

Let F and F' be two foliations of $\mathbb{R}^n$ by hyperplanes perperdicular to a principal axis θ not necessarily the same. We say that F and F' are *equally distributed* if there is a homeomorphism $D : \mathbb{R}^n \to \mathbb{R}^n$ that maps each leave of F into a leave of F' with the identical classification as solution of the bi-dimensional problem.

Lemma 3. *For any isomorphism φ and any foliation F_θ of S, F_θ and $\varphi(F_\theta)$ are equally distributed.*

Proof. We can assume without loss of generality that $\theta = x$. Consider the expressions $P(x, y, z) = (b_1x+b_0)+(b_4x+b_2)y+(b_5x+b_3)z+(b_7x+b_6)yz$ and $P\circ\varphi^{-1}(x, y, z) = (B_1x+B_0)+(B_4x+B_2)y+(B_5x+B_3)z+(B_7x+B_6)yz$. By virtue of Lemma 2, all the leaves of F_x, except at most for one, have the same classification as solutions of a bi-dimensional case. Suppose that all the leaves of F_x are hyperbolas. Then by virtue of Lemma 1, b_7x+b_6 is non-null for any value of x. This implies that $b_7 = 0$ and $b_6 \neq 0$. Using Equation (1.9) we have that $B_7 = 0$ and Equation (1.8) ensures that $B_6 \neq 0$. So, all the leaves of $\varphi(F_x)$ are hyperbolas. Now suppose that all the leaves of F_x, except for one, are hyperbolas. This implies that there exists a unique value of x_0 of x for which b_7x+b_6 vanishes. So, b_7 is non-null. Using again Equation (1.9), we get that B_7 is non-null and hence all the leaves of $\varphi(F_x)$ are hyperbolas, except for exactly one of them. In view of Lemma 1, we have three possibilities for this distinct leave: a straight line, a plane, and the empty set. If the distinct leave of F_x is a straight line, then by virtue of Lemma 1 the coefficients b_4x+b_2 and $b_5x + b_3$ do not vanish simultaneously at x_0. This means that at least one of the discriminants $b_7b_2 - b_6b_4$ and $b_7b_3 - b_6b_5$ is no-null. Since $B_7B_2 - B_6B_4 = ab^2c(b_7b_2 - b_6b_4)$ and that $B_7B_3 - B_6B_5 = abc^2(b_7b_3 - b_6b_5)$, at least one of these determinants is non-null and we can conclude that the distinct leave of $\varphi(F_x)$ is a straight line. Now, if the distinct leave of F_x is a plane, the two previous determinants are null at the same time, and in addition, the determinant $b_7b_0 - b_6b_1$ needs to be zero. In view of the previous comments, it suffices to prove that $b_7b_0 - b_6b_1 = 0$ if and only if $B_7B_0 - B_6B_1 = 0$. But, $B_7B_0 - B_6B_1 = abc[(b_7b_0 - b_6b_1) + e(b_7b_2 - b_6b_4) + f(b_7b_3 - b_6b_5)]$ and hence vanishes. So, the distinct leave of $\varphi(F_x)$ is also a plane. If the distinct leave of F_x is the empty set, then the only difference with the case of the plane is that the discriminant $b_7b_0 - b_6b_1$ is non-null. Using again the formula for $B_7B_0 - B_6B_1$ we can conclude that it is also non-null. Hence the distinct leave of $\varphi(F_x)$ is the empty set. The remaining possibilities are: (a) all the leaves of F_x being straight lines (b,c) all the leaves of F_x being straight lines, except for one that is: (b) a plane (c) the empty set, (d) all the leaves of F_x being the empty set, (f) all the leaves of F_x being the empty set, except for one that is a plane, and (g) all the leaves of F_x being planes. In all these cases the proof follows in a similar way to those we exhibited. □

Here are some immediate consequences of the previous Lemma.

Corollary 1. *The distribution of the foliation F_θ of S is an invariant.*

Corollary 2. *The combination of distributions of the three foliations F_x, F_y and F_z of S is an invariant.*

A close examination to the proof of Lemma 3 shows that the determinants $b_1b_7 - b_4b_5$, $b_2b_7 - b_4b_6$, and $b_3b_7 - b_5b_6$ play a fundamental role in determining the distribution of leaves of the foliations F_θ. We also saw that the corresponding determinants of the foliations $\varphi(F_\theta)$ are related with the formers by the formulas: $B_1B_7 - B_4B_5 = a^2bc(b_1b_7 - b_4b_5)$, $B_2B_7 - B_4B_6 = ab^2c(b_2b_7 - b_4b_6)$, and $B_3B_7 - B_5B_6 = abc^2(b_3b_7 - b_5b_6)$. Thus, we get the following invariants.

Corollary 3.

1. *$b_1b_7 - b_4b_5$ vanishes if and only if so does $B_1B_7 - B_4B_5$.*
2. *$b_2b_7 - b_4b_6$ vanishes if and only if so does $B_2B_7 - B_4B_6$.*
3. *$b_3b_7 - b_5b_6$ vanishes if and only if so does $B_3B_7 - B_5B_6$.*
4. *$\mathrm{sg}((b_1b_7 - b_4b_5)(b_2b_7 - b_4b_6)(b_3b_7 - b_5b_6)) = \mathrm{sg}((B_1B_7 - B_4B_5)(B_2B_7 - B_4B_6)(B_3B_7 - B_5B_6))$.*

The following lemma establishes conditions under which we can cancel some coefficients of P in order to obtain a much simpler equivalent expression.

Lemma 4. *Let $P(x, y, z) = b_0 + b_1x + b_2y + b_3z + b_4xy + b_5xz + b_6yz + b_7xyz$. Then,*

1. *if $b_7 \neq 0$, then for any non-null values of a, b and c, the isomorphism*

$$\varphi : (x, y, z) \mapsto \left(ax - \frac{b_6}{b_7}, by - \frac{b_5}{b_7}, cz - \frac{b_4}{b_7}\right)$$

leads to $P \circ \varphi(x, y, z) = B_0 + B_1x + B_2y + B_3z + B_7xyz$, and

2. *if $b_7 = 0$ and b_4, b_5 and b_6 are non-null, then for any non-null values of a, b and c, the isomorphism*

$$\varphi : (x, y, z) \mapsto$$
$$\left(ax + \frac{b_1b_6 - b_2b_5 - b_3b_4}{2b_4b_5}, by + \frac{b_2b_5 - b_1b_6 - b_3b_4}{2b_4b_6}, cz + \frac{b_3b_4 - b_1b_6 - b_2b_5}{2b_5b_6}\right)$$

leads to $P \circ \varphi(x, y, z) = B_0 + B_4xy + B_5xz + B_6yz + B_7xyz$.

Proof. The result is obtained by equating to zero the expressions of B_4, B_5, and B_6, a linear system in the variables d, e, and f is obtained. Solving this system get (1). Statement (2) follows analogously from solving the linear system obtained by equating to zero the expressions of B_1, B_2, and B_3. $\square$

1.2 Foliation classification

We now focus on summarizing the information provided by the results obtained so far by describing in detail all the possibilities for the foliations F_x of S.

P1 For all real value of x, F_x is empty. In this instance $b_0 \neq 0$ and the remaining coefficients are all zero. (This leads to the empty set as the unique solution.)

P2 For all real values of x, F_x is a straight line. This case emerges when $b_6 = b_7 = 0$, and the linear system $\begin{cases} b_4x + b_2 = 0 \\ b_5x + b_3 = 0 \end{cases}$ is incompatible. In this case we have that the general expression of P is $P(x, y, z) = b_0 + b_1x + b_2y + b_3z + b_4xy + b_5xz$, with $b_2b_5 - b_3b_4 \neq 0$.

P3 For all real values of x, F_x is a hyperbola. In this case, $b_7 = 0$ and $b_6 \neq 0$. The general expression of P in this case is $P(x, y, z) = b_0 + b_1x + b_2y + b_3z + b_4xy + b_5xz + b_6yz$, with $b_6 \neq 0$. Here we distinguish three non-equivalent subcases by attending to the number of degenerated hyperbolas that appear. We saw in Lemma 1 that the number of such hyperbolas coincides with the number of real roots of the polynomial $b_4b_5x^2 + (b_3b_4 + b_2b_5 - b_1b_6)x + (b_2b_3 - b_0b_6)$.

P4 For all real values of x, F_x is a plane. This occurs when all of the coefficients are zero. (On this occasion, the resulting space is the whole space $\mathbb{R}^3$.)

P5 There is a real number a such that for all real value of $x \neq a$, F_x is empty and F_a is a plane. This case appears when $b_1 \neq 0$ and $b_i = 0$ for all $2 \leq i \leq 7$. (There is no restriction on b_0.)

P6 There is a real number a such that for all real value of $x \neq a$, F_x is a straight line and F_a is empty. This case arises when $b_6 = b_7 = 0$, at least one of the coefficients b_4 and b_5 is non-null, and the linear system $\begin{cases} b_4x + b_2 = 0 \\ b_5x + b_3 = 0 \end{cases}$ is compatible, but its solution does not satisfies the equation $b_1x + b_0 = 0$. Thus, we have that the general expression of P is $P(x, y, z) = b_0 + b_1x + \alpha b_4y + \alpha\beta b_4z + b_4xy + \beta b_4xz$ whenever $b_4 \neq 0$. If $b_4 = 0$, then we have the expression $P(x, y, z) = b_0 + b_1x + \alpha b_5z + b_5xz$.

P7 There is a real number a such that for all real value of $x \neq a$, F_x is a hyperbola and F_a is empty. This cases emerges when $b_7 \neq 0$,

Table 1.2: Possibility tests.

<table>
<tr><th>Poss.</th><th>b_0</th><th>b_1</th><th>b_2</th><th>b_3</th><th>b_4</th><th>b_5</th><th>b_6</th><th>b_7</th><th>$b_7b_0 - b_6b_1$</th><th>$b_7b_2 - b_6b_4$</th><th>$b_7b_3 - b_6b_5$</th><th>$b_5b_2 - b_4b_3$</th><th>$b_4b_0 - b_2b_1$</th><th>$b_5b_0 - b_3b_1$</th></tr>
<tr><td>P1</td><td>$\neq 0$</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>P2</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>0</td><td>0</td><td>0</td><td>0</td><td>$\neq 0$</td><td>0</td><td>0</td></tr>
<tr><td>P3</td><td></td><td></td><td></td><td></td><td></td><td></td><td>$\neq 0$</td><td>0</td><td></td><td></td><td></td><td></td><td></td><td>0</td></tr>
<tr><td>P4</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>P5</td><td></td><td>$\neq 0$</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td rowspan="3">P6</td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3">0</td><td rowspan="3">0</td><td>$\neq 0$</td><td>0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td><td>$\neq 0$</td><td>0</td></tr>
<tr><td>0</td><td>$\neq 0$</td><td>0</td><td>$\neq 0$</td></tr>
<tr><td>$\neq 0$</td><td>$\neq 0$</td><td>$\neq 0$</td><td>$\neq 0$</td></tr>
<tr><td>P7</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>$\neq 0$</td><td>$\neq 0$</td><td>0</td><td>0</td><td>0</td><td>$\neq 0$</td><td></td></tr>
<tr><td rowspan="3">P8</td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td>$\neq 0$</td><td>0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td><td rowspan="3">0</td></tr>
<tr><td>0</td><td>$\neq 0$</td></tr>
<tr><td>$\neq 0$</td><td>$\neq 0$</td></tr>
<tr><td rowspan="3">P9</td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3">$\neq 0$</td><td rowspan="3"></td><td>0</td><td>$\neq 0$</td><td>$\neq 0$</td><td rowspan="3"></td><td rowspan="3"></td></tr>
<tr><td>$\neq 0$</td><td>0</td><td>$\neq 0$</td></tr>
<tr><td>$\neq 0$</td><td>$\neq 0$</td><td>0</td></tr>
<tr><td>P10</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>$\neq 0$</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
</table>

$\begin{cases} b_7x + b_6 = 0 \\ b_4x + b_2 = 0 \\ b_5x + b_3 = 0 \end{cases}$ is compatible, but its solution does not satisfies the equation $b_1x + b_0 = 0$. Thus, we have that there exist real numbers α, β, and γ such that $P(x, y, z) = b_0 + b_1x + \alpha\gamma b_7y + \alpha\beta b_7z + \gamma b_7xy + \beta b_7xz + \alpha b_7yz + b_7xyz$, with $b_0 + \alpha b_1 \neq 0$. From this, taking $b_7 = 1$ and using Lemma 4 (a), we obtain that $P(x, y, z) = B_0 + B_1x + xyz$, with $B_0 = \frac{1}{abc}(b_0 + \alpha b_1) \neq 0$ and $B_1 = \frac{1}{bc}(b_1 - \gamma\beta)$, is a simplified general expression of P.

P8 There is a real number a such that for all real value of $x \neq a$, F_x is a straight line and F_a is a plane. This instance appears when $b_6 = b_7 = 0$, at least one of the coefficients b_4 and b_5 is non-null, and the linear system $\begin{cases} b_1x + b_0 = 0 \\ b_4x + b_2 = 0 \\ b_5 + b_3 = 0 \end{cases}$ is compatible. We thus get that the general expression of P is $P(x, y, z) = \alpha\beta b_4 + \beta b_4x + \alpha b_4y + \alpha\gamma b_5z + b_4xy + \gamma b_4yz$ whenever $b_4 \neq 0$. If $b_4 \neq 0$, then $P(x, y, z) = \alpha\beta b_5 + \beta b_5x + \alpha b_5z + b_5xz$.

P9 There is a real number a such that for all real value of $x \neq a$, F_x is a hyperbola and F_a is a straight line. This case occurs when $b_7 \neq 0$, and $\begin{cases} b_4x + b_2 = 0 \\ b_5x + b_3 = 0 \\ b_7x + b_6 = 0 \end{cases}$ is incompatible. We thus have that the general expression of P is $P(x, y, z) = b_0 + b_1x + b_2y + b_3z + b_4xy + b_5xz + b_6yz + b_7xyz$, where $b_2b_7 - b_4b_6$ and $b_3b_7 - b_5b_6$ are non-null. Using Lemma 4 (a) and taking $b_7 = 1$, this expression becomes $P(x, y, z) = B_0 + B_1x + B_2y + B_3z + xyz$, with $B_0 = \frac{1}{abc}(b_0 - b_1b_6 - b_2b_5 - b_3b_4 + b_4b_5b_6)$, $B_1 = \frac{1}{bc}(b_1 - b_4b_5)$, $B_2 = \frac{1}{ac}(b_2 - b_4b_6)$, and $B_3 = \frac{1}{ab}(b_3 - b_5b_6)$. In this case we have three subcases depending on the number of real roots of the polynomial $(b_1b_7 - b_4b_5)x^2 + (b_1b_6 + b_0b_7 - b_3b_4 - b_2b_5)x + (b_0b_6 - b_2b_3)$.

P10 There is a real number a such that for all real value of $x \neq a$, F_x is a hyperbola and F_a is a plane. This case arises when $b_7 \neq 0$, and $\begin{cases} b_1x + b_0 = 0 \\ b_4x + b_2 = 0 \\ b_5x + b_3 = 0 \\ b_7x + b_6 = 0 \end{cases}$ is compatible. So, the general expression of P is $P(x, y, z) = \alpha\beta b_7 + \beta b_7x + \alpha\gamma b_7y + \alpha\delta b_7z + \gamma b_7xy + \delta b_7xz + \alpha b_7yz + b_7xyz$. Using Lemma 4 (a) and taking $b_7 = 1$, we obtain the simplified expression $P(x, y, z) = B_1x + B_2y + xyz$, with $B_1 = \frac{1}{bc}(\alpha\beta - \gamma\delta)$ and $B_2 = \frac{1}{ac}(\beta - \alpha\gamma)$.

Observe that the study of the foliation F_x allowed us to make a preliminary classification of the solutions and in most cases we attained to reduce the number of parameters to be analyzed. Obviously, the possibilities for F_y and F_z are exactly the same as for F_x. The difference lies in the conditions that each possibility imposes on the coefficients of P, which can easily obtained by applying the respective coefficient correspondence listed in Table 3 to each row in Table 1.2.

We can produce a closer classification to the marching cube's cases if instead of one only foliation, we consider the combinations of foliations F_x, F_y and F_z. Combining the tables for F_x, F_y, and F_z, we get the following list.

C1 $(P1, P1, P1)$. This occurs when $b_0 \neq 0$ and the remaining coefficients are all null. Thus, the general expression for P is $P(x, y, z) = 1$.

C2 $(P2, P2, P2)$. This combination emerges when $b_4 = b_5 = b_6 = b_7 = 0$ and at least two of the coefficients b_1, b_2 and b_3 are not null. So, the general expression for P is $P(x, y, z) = b_0 + b_1x + b_2y + b_3z$.

C3 $(P2, P2, P3)$. This happens when b_3 and b_4 are non-null and $b_5 = b_6 = b_7 = 0$. Thus, in general, $P(x, y, z) = b_0 + b_1x + b_2y + b_3z + b_4xy$. Since the polynomial $b_3b_4z + (b_0b_4 - b_1b_2)$ has always only a (real) root, F_z always contains a only degenerated hyperbola.

C4 $(P2, P2, P5)$. This combination arises when $b_3 \neq 0$ and the remaining coefficients are null. The general expression of P in this case is $P(x, y, z) = z$

C5 $(P2, P3, P3)$. This combination appears when the coefficients b_4 and b_5 are non-null, $b_6 = b_7 = 0$, and $b_2b_5 - b_3b_4$ is not zero. The general expression of P in this case is $P(x, y, z) = b_0 + b_1x + b_2y + b_3z + b_4xy + b_5xz$, with $b_2b_5 - b_3b_4$ non-null. Because the polynomial $b_3b_4z - (b_0b_4 - b_1b_2)$ has always only a (real) root, F_z always contains only one degenerated hyperbola.

C6 $(P3, P3, P3)$. This occurs when the coefficients b_4, b_5 and b_6 are non-null and $b_7 = 0$. Here, the general expression of P is $P(x, y, z) = b_0 + b_1x + b_2y + b_3z + b_4xy + b_5xz + b_6yz$. Using Lemma 4 we obtain a simplified expression $P(x, y, z) = B_0 + B_4xy + B_5xz + B_6yz$, with $B_0 = \frac{(b_3b_4)^2+(b_1b_6)^2+(b_2b_5)^2-2b_1b_3b_4b_5-2b_2b_3b-4b_5-2b_1b_2b_5b_6+4b_0b_4b_5b_6}{4b_4b_5b_6}$, $B_4 = abb_4$, $B_5 = acb_5$, and $B_6 = bcb_6$. Here we have three subcases: (a) each one of the foliations F_x, F_y, and F_z contains two degenerated hyperbolas, (b) each one of the foliations F_x, F_y, and F_z contains one degenerated hyperbolas, (c) none of the foliations F_x, F_y, and F_z contains degenerated hyperbolas. The subcases (a), (b), and (c) appear when the polynomial $B_4B_5x^2 - B_0B_6$ has two, one, and none real roots, respectively. (The reader must have noticed that the polynomial $B_4B_5x^2 - B_0B_6$ corresponds to the degenerated hyperbolas of F_x. The polynomials corresponding to the foliations F_y and F_z are $B_4B_6y^2 - B_0B_5$ and $B_5B_6y^2 - B_0B_4$, respectively. However, all these polynomials have exactly the same discriminant.)

C7 $(P3, P3, P6)$. This combination appears when $b_4 = b_7 = 0$, b_6 and b_5 are non-null, and there are real numbers α and β such that $b_1 = \alpha\beta b_6$, $b_2 = \alpha b_6$, $b_5 = \beta b_6$, and $b_0 \neq \alpha b_3$. Thus, the general expression of P is $P(x, y, z) = b_0 + \alpha\beta b_6 x + \alpha b_6 y + b_3 z + \beta b_6 xz + b_6 yz$. In this case both F_x and F_y do not contain any degenerated hyperbola. The reason for this is that the polynomials corresponding to the degenerated hyperbolas of F_x and F_y are $b_6(b_0 - \alpha b_3)$ and $\beta b_6(b_0 - \alpha b_3)$, respectively, and both are non-null constants.

C8 $(P3, P3, P8)$. This combination emerges when $b_4 = b_7 = 0$ and there are real numbers α, β and γ such that $b_0 = \alpha\gamma b_6$, $b_1 = \alpha\beta b_6$, $b_2 = \alpha b_6$, $b_3 = \gamma b_6$, and $b_5 = \beta b_6$. Thus, the general expression of P is $P(x, y, z) = \alpha\gamma b_6 + \alpha\beta b_6 x + \alpha b_6 y + \gamma b_6 z + \beta b_6 xz + b_6 yz$. Here, all the leaves of F_x and F_y are degenerated hyperbolas. The reason for this is that the polynomials corresponding to the degenerated hyperbolas of F_x and F_y have all their coefficients null.

C9 $(P3, P6, P6)$. This combination occurs when $b_1 = b_4 = b_5 = b_7 = 0$, $b_6 \neq 0$, and there is a real numbers α and β such that $b_2 = \alpha b_6$, $b_3 = \beta b_6$, and $b_0 \neq \alpha\beta b_6$. Thus, the expression of P takes the form $P(x, y, z) = b_0 + \alpha b_6 y + \beta b_6 z + b_6 yz$, with $b_0 \neq \alpha\beta b_6$. Since the polynomial corresponding to the degenerated hyperbolas of F_x is a the constant $b_6(b_0 - \alpha\beta b_6) \neq 0$, we can conclude that none leave of F_x is a degenerated hyperbola.

C10 $(P3, P8, P8)$. This combination arises when $b_1 = b_4 = b_5 = b_7 = 0$, $b_6 \neq 0$, and there are real numbers α and β such that $b_2 = \alpha b_6$, $b_3 = \beta b_6$, and $b_0 = \alpha\beta b_6$. So, the general expression of P in this case is $P(x, y, z) = \alpha\beta b_6 + \alpha b_6 y + \beta b_6 z + b_6 yz$. Obviously, all the leaves of F_x are degenerated hyperbolas.

C11 $(P4, P4, P4)$. This combination only appears when $P \equiv 0$.

C12 $(P7, P7, P7)$. This combination appears when $b_7 \neq 0$ and there exist real numbers α, β, and γ such that $b_1 = \beta\gamma b_7$, $b_2 = \alpha\gamma b_7$, $b_3 = \alpha\beta b_7$, $b_4 = \gamma b_7$, $b_5 = \beta b_7$, $b_6 = \alpha b_7$, and $b_0 \neq \alpha\beta\gamma b_7$. This forces $P(x, y, z) = b_0 + \beta\gamma b_7 x + \alpha\gamma b_7 y + \alpha\beta b_7 z + \gamma b_7 xy + \beta b_7 xz + \alpha b_7 yz + b_7 xyz$, with $b_0 \neq \alpha\beta\gamma b_7$. Using Lemma 4 (a) and putting $b_7 = 1$, we obtain the simplified expression $P(x, y, z) = B_0 + xyz$, with $B_0 = \frac{1}{abc}(b_0 - \alpha\beta\gamma)$. From this expression it follows immediately that all the foliations F_x, F_y, and F_z contain exactly one degenerated hyperbola.

C13 $(P7, P9, P9)$. This combination emerges when $b_7 \neq 0$ and there exist real numbers α, β, and γ such that $b_2 = \alpha\gamma b_7$, $b_3 = \alpha\beta b_7$, $b_4 = \gamma b_7$, $b_5 = \beta b_7$, $b_6 = \alpha b_7$, $b_1 \neq \beta\gamma b_7$, and $b_0 \neq \alpha b_1$. This gives, $P(x, y, z) = b_0 + b_1 x + \alpha\gamma b_7 y + \alpha\beta b_7 z + \gamma b_7 xy + \beta b_7 xz + \alpha b_7 yz + b_7 xyz$, with $b_1 \neq \beta\gamma b_7$, and $b_0 \neq \alpha b_1$. Using Lemma 4(a) and setting $b_7 = 1$ we obtain the simplified expression $P(x, y, z) = B_0 + B_1 x + xyz$, with $B_0 = \frac{1}{abc}(b_0 - \alpha b_1)$ and $B_1 = \frac{1}{bc}(b_1 - \gamma\beta)$. From this expression we can easily see that F_x always has two degenerated hyperbolas and F_y and F_z only have one.

C14 $(P9, P9, P9)$. This combination arises when $b_7 \neq 0$ and at least two of the numbers $b_1 b_7 - b_4 b_5$, $b_2 b_7 - b_4 b_6$, and $b_3 b_7 - b_5 b_6$ are non-null. From Lemma 4 (a) and setting $b_7 = 1$, we get the simplified expression $P(x, y, z) = B_0 + B_1 x + B_2 y + B_3 z + xyz$, with $B_0 = \frac{1}{abc}(b_0 - b_1 b_6 - b_2 b_5 - b_3 b_4 + 2 b_4 b_5 b_6)$, $B_1 = \frac{1}{bc}(b_1 - b_4 b_5)$, $B_2 = \frac{1}{ac}(b_2 - b_4 b_6)$, and $B_3 = \frac{1}{ab}(b_3 - b_5 b_6)$. In this case we have six subcases: (a) $B_1 B_2 B_3 < 0$, and each foliation F_x, F_y, and F_z has exactly two degenerated hyperbolas, (b) $B_1 B_2 B_3 < 0$, and each one of the foliations F_x, F_y and F_z has exactly one degenerated hyperbola, (c) $B_1 B_2 B_3 < 0$, and none of the foliations F_x, F_y, and F_z contains degenerated hyperbolas, (d) $B_1 B_2 B_3 = 0$, the foliation F_x does not contain degenerated hyperbolas, and each one of the foliations F_y, and F_z contains exactly two degenerated hyperbolas (e) $B_1 B_2 B_3 = 0$, the foliation F_x contains exactly one degenerated hyperbola, and each one of the foliations F_y, and F_z has exactly two degenerated hyperbolas (f) $B_1 B_2 B_3 > 0$ and each of the foliations F_x, F_y, and F_z contains exactly two degenerated hyperbolas.

C15 $(P9, P9, P10)$. This combination appears when $b_7 \neq 0$ and there are real numbers α, β, γ, and δ such that $b_0 = \alpha\delta b_7$, $b_1 = \alpha\gamma b_7$, $b_2 = \alpha\beta b_7$, $b_3 = \delta b_7$, $b_4 = \alpha b_7$, $b_5 = \gamma b_7$, $b_6 = \beta b_7$, and $\delta - \beta\gamma \neq 0$. Thus, the general expression of P is $P(x, y, z) = \alpha\delta b_7 + \alpha\gamma b_7 x + \alpha\beta b_7 y + \delta b_7 z + \alpha b_7 xy + \gamma b_7 xz + \beta b_7 yz + b_7 xyz$, with $\delta - \beta\gamma \neq 0$. From Lemma 4 (a) and taking $b_7 = 1$, we deduce the simplified expression $P(x, y, z) = B_3 z +$

xyz, with $B_3 = \frac{1}{ab}(\delta - \beta\gamma)$. Obviously, all the hyperbolas contained in any of the foliations F_x, F_y, and F_z, are degenerated.

C16 $(P10, P10, P10)$. This combination appears when $b_7 \neq 0$ and there are real numbers α, β and δ such that $b_0 = \alpha\delta b_7$, $b_1 = \alpha\beta\delta b_7$, $b_2 = \alpha\beta b_7$, $b_3 = \delta b_7$, $b_4 = \alpha b_7$, $b_5 = \beta\delta b_7$, and $b_6 = \beta b_7$. Thus, the general expression of P is $P(x, y, z) = \alpha\delta b_7 + \alpha\beta\delta b_7 x + \alpha\beta b_7 y + \delta b_7 z + \alpha b_7 xy + \beta\delta b_7 xz + \beta b_7 yz + b_7 xyz$. Using Lemma 4 (a) and putting $b_7 = 1$, we obtain the simplified expression $P(x, y, z) = xyz$. This clearly forces any hyperbola contained in any of the foliations F_x, F_y, and F_z to be degenerated.

All the other combinations are impossible since they have opposite nullity test on some of the entries of Table 1.2.

1.3 The complete classification

The list above allows us to exhibit the more simple representative for each class of our classification. As the following theorem shows, we have infinite of such classes — only one for each combination different to $(P3, P3, P3)$ and $(P9, P9, P9)$, three for the combination $(P3, P3, P3)$, and infinite for the combination $(P9, P9, P9)$.

Theorem 1. *Let S be the solution of the linear interpolation equation. Then, S is equivalent to one and only one solution of the following equations:*

C1 *Empty set.*

C2 $(P2, P2, P2)$, $x + y + z = 0$.

C3 $(P2, P2, P3)$, $xy + z = 0$.

C4 $(P2, P2, P5)$, $z = 0$.

C5 $(P2, P3, P3)$, $xy + xz + z = 0$.

C6(a) $(P3, P3, P3)$, $xy + yz + xz - 1 = 0$.

C6(b) $(P3, P3, P3)$, $xy + yz + xz = 0$.

C6(c) $(P3, P3, P3)(c)$, $xy + yz + xz + 1 = 0$.

C7 $(P3, P3, P6)$, $xz + yz + 1 = 0$.

C8 $(P3, P3, P8)$, $xz + yz = 0$.

C9 $(P3, P6, P6)$, $yz + 1 = 0$.

C10 $(P3, P8, P8)$, $yz = 0$.

C11 $\mathbb{R}^3$

C12 $(P7, P7, P7)$, $xyz + 1 = 0$.

C13 $(P7, P9, P9)$, $xyz - x + 1 = 0$.

C14(a) $(P9, P9, P9)$, $xyz - x - y - z + B_0 = 0$, *with* $B_0 \in \mathbb{R}$. *(There is an equivalence class for each non-negative real value of* B_0*.)*

C14(b) $(P9, P9, P9)$, $xyz - y - z = 0$.

C14(c) $(P9, P9, P9)$, $xyz - y - z + 1 = 0$.

C14(d) $(P9, P9, P9)$, $xyz + x + y + z + B_0 = 0$, *with* $B_0 \in \mathbb{R}$. *(There is an equivalence class for each non-negative real value of* B_0*.)*

C15 $(P9, P9, P10)$, $xyz - z = 0$.

C16 $(P10, P10, P10)$, $xyz = 0$.

Proof. The proof of each item consists in taking the simplified expression of P given in the list above and finding an equivalence $\varphi(x, y, z) = (ax + d, by + e, cz + f)$ such that $P \circ \varphi$ coincides with the expression P' presented in the statement of this theorem. To achieve this, we set each coefficient of $P \circ \varphi$ equal to its corresponding in P' and solve the resulting system of linear equations to determine values for the parameters a, b, c, d, e, and f. To illustrate the procedure, we shall do in detail the case of $(P2, P2, P2)$. The remaining cases are purely analogous.

In view of the list of combination, we have that the expression for the solutions whose combination of foliations is $(P2, P2, P2)$ is $P(x, y, z) = b_0 + b_1x + b_2y + b_3z = 0$, where b_1, b_2 and b_3 are non-null. Thus, $P \circ \varphi(x, y, z) = (b_0 + cb_1 + eb_2 + fb_3) + ab_1x + bb_2y + cb_3z$. Since the expression of P' is $P'(x, y, z) = x + y + z$, we obtain the system: $ab_1 = 1$, $bb_2 = 1$, $cb_3 = 1$, and $b_0 + cb_1 + eb_2 + fb_3 = 0$, which has infinite solutions. Therefore, the respective solutions of $P(x, y, z) = 0$ and $P'(x, y, z) = 0$ are equivalent. □

The existence of infinitely many equivalence classes is due to the fact the combination $(P9, P9, P9)$ contains two general solutions depending on a real parameter B_0, namely (**C14(a)**): $xyz - x - y - z + B_0 = 0$, and

(**C14(d)**): $xyz + x + y + z + B_0 = 0$. We conclude this section by showing that, although the solutions of these equations corresponding to different non-negative values of the parameter B_0 are non-equivalent in the sense we defined at the beginning of the chapter, all them are grouped into a quite small number of classes of diffeomorphic solutions.

Theorem 2. *1. Let S and S' be the solutions of the equation $xyz - x - y - z + B_0 = 0$ obtained from two different non-negative values of B_0. If these values are together in one of the regions $B_0 < 2$, $B_0 = 2$, and $B_0 > 2$, then S and S' are difeomorphic.*

2. Let S and S' be the solutions of the equation $xyz + x + y + z + B_0 = 0$ obtained from two different non-negative values of B_0. Then, S and S' are diffeomorphic.

The proof of this theorem is an immediate consequence of the following result of Morse Theory (see [26]).

Definition 1. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a C^∞-function.

1. The point $p \in \mathbb{R}^3$ is said to be a critical point of F if $F_x(p) = F_y(p) = F_z(p) = 0$. If p is a critical point of F, then $F(p)$ is called critical value of F.

2. The Hessian matrix $\mathcal{H}(F, p)$ of F at the point p is defined by:

$$\mathcal{H}(F,p) = \begin{pmatrix} F_{xx}(p) & F_{xy}(p) & F_{xz}(p) \\ F_{yx}(p) & F_{yy}(p) & F_{yz}(p) \\ F_{zx}(p) & F_{zy}(p) & F_{zz}(p) \end{pmatrix}.$$

3. A critical point p of F for which $\mathcal{H}(F, p)$ is invertible is called non-degenerated. If all the critical points of F are non-degenerated, we say that F is a Morse function.

Lemma 5. *Let $a < b$ be real numbers such that the interval $[a, b]$ does not contains critical values of F. Then the surfaces $F^{-1}(\{a\})$ and $F^{-1}(\{b\})$ are diffeomorphic.*

Proof. (of the Theorem 2)

1. Consider the function $F(x, y, z) = xyz - x - y - z$. It is easy to verify that the critical points of F are $(1, 1, 1)$ and $(-1, -1, -1)$, which are non-degenerated. Since the regular values of F are -2 and 2, the result follows immediately from Lemma 5.

2. Consider the function $F(x, y, z) = xyz+x+y+z$. It can be easily seem that F has no critical points and hence has no critical values. Using again Lemma 5, we obtain the result.

□

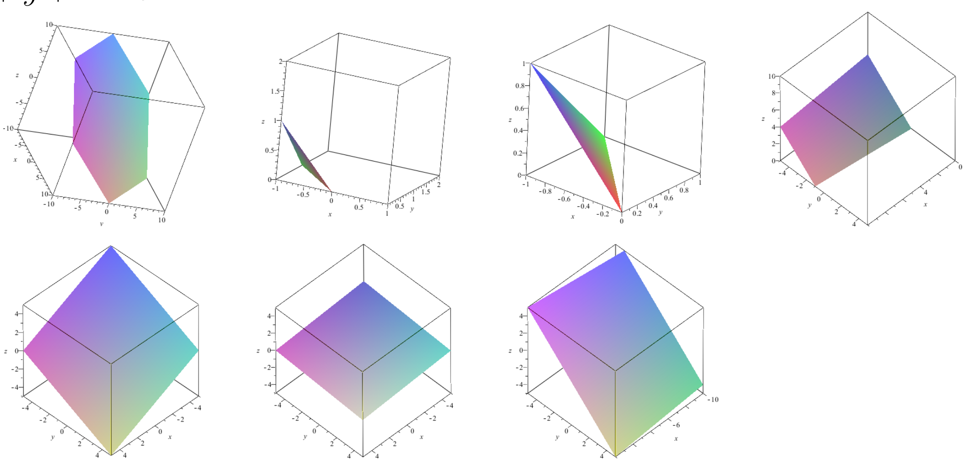
As we mentioned before, Theorem 1 was obtained previously by Nielson his work [2]. However, the main contributions of the chapter lie in the approach via foliations which provides some geometrical insight to the question. As we shall see later, the intuition we gain here will be of great importance in the treatment of the problem in dimension four.

1.4 Intersections of general solutions with a cube

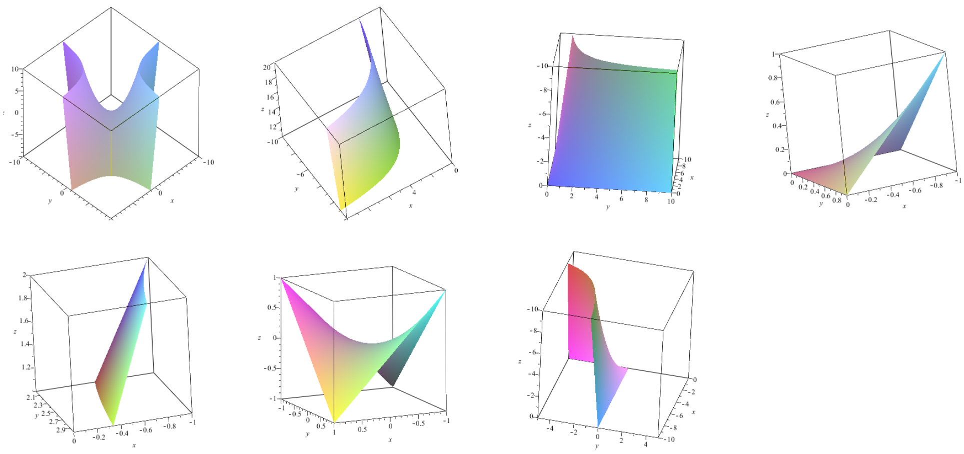
The following list presentes some illustrative examples of the intersections of each general solution of the equation $f(x, y, z) = 0$ with the cube $[0, 1]^3$. This provides some intuition of the configurations of marching cubes.

C1 This combination only returns the empty cube.

C2 $x + y + z = 0$

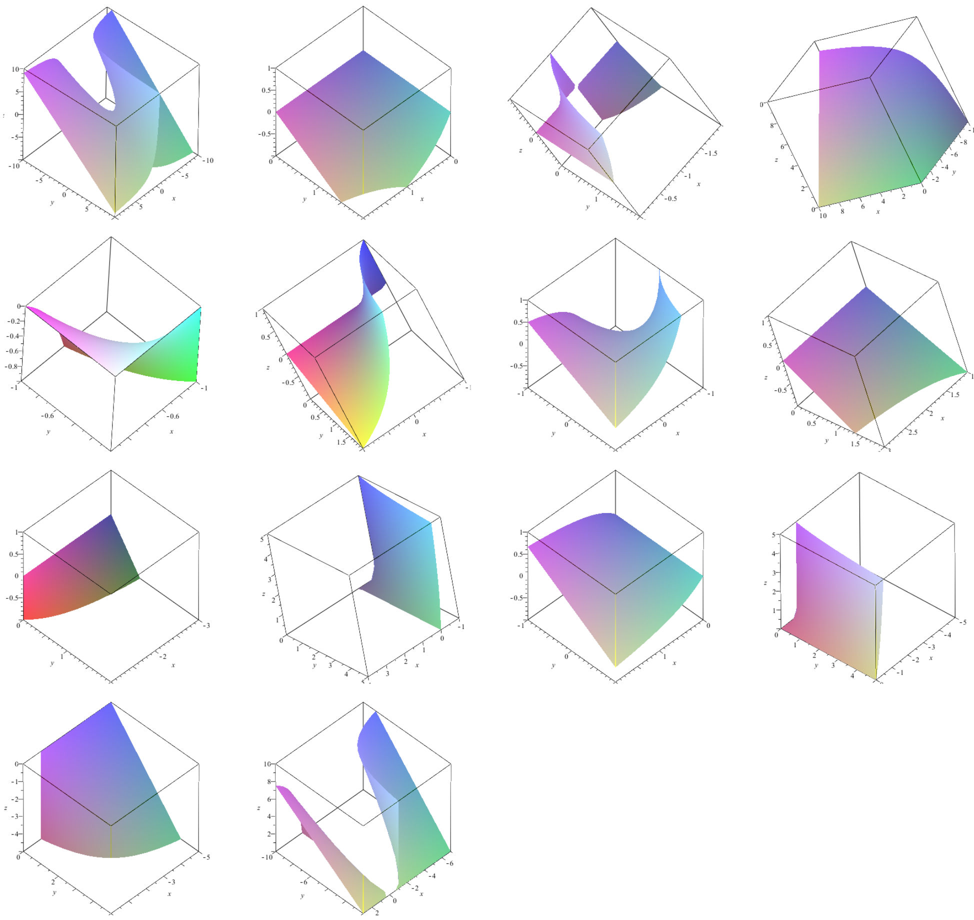


C3 $xy + z = 0$

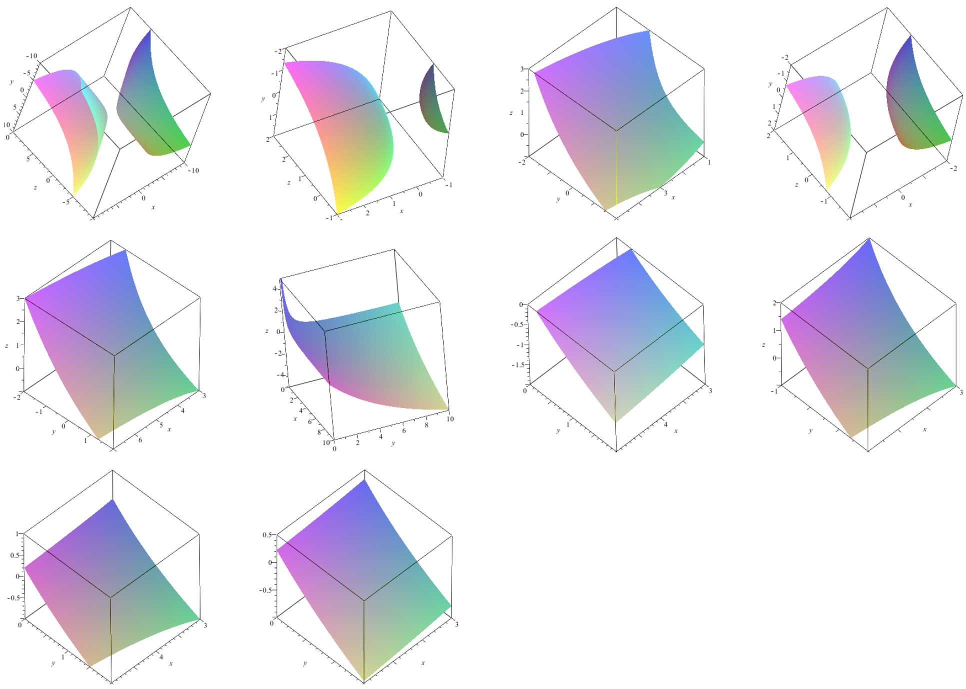


C4 This combination only returns the cube with a filled face.

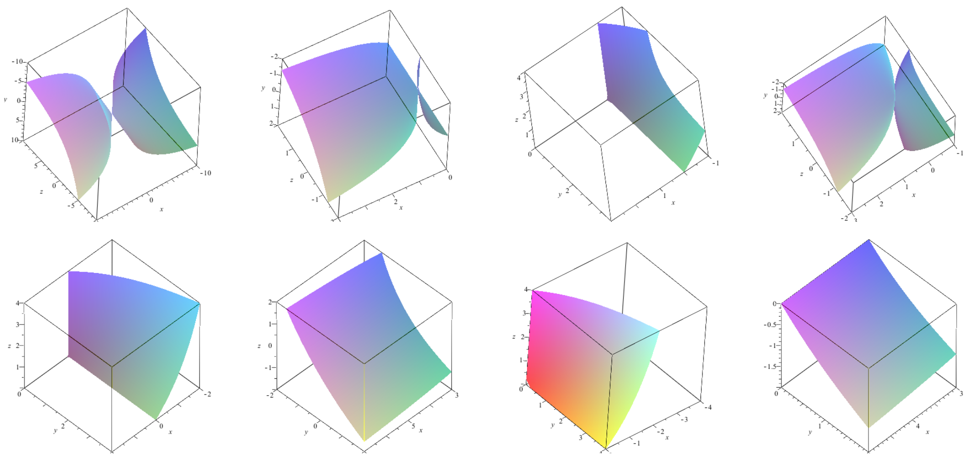
C5 $xy + xz + z = 0$



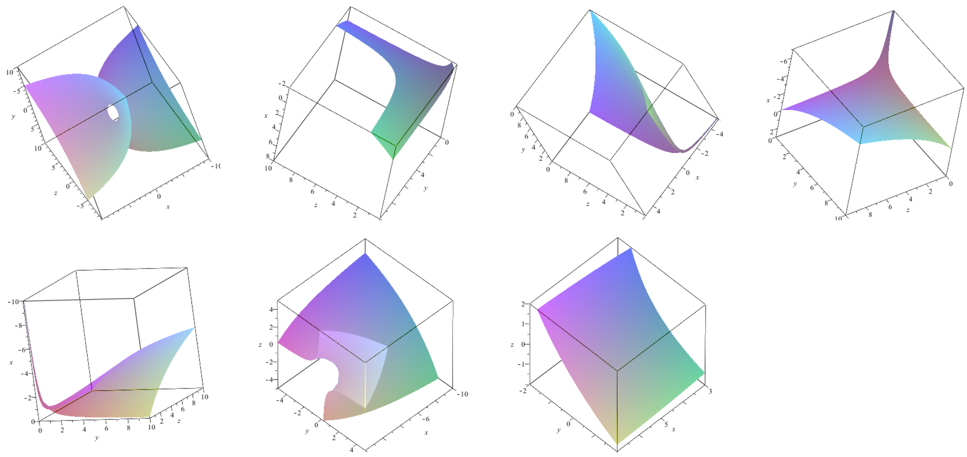
C6(a) $xy + yz + xz - 1 = 0$



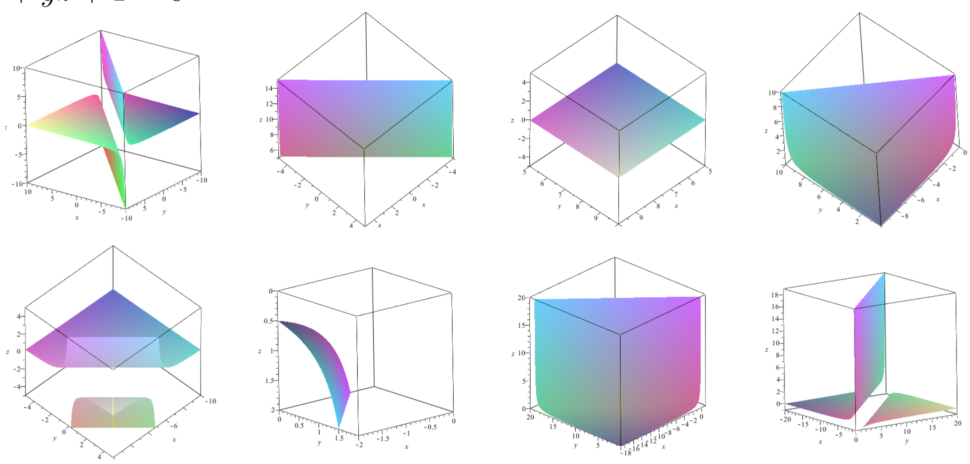
C6(b) $xy + yz + xz = 0$



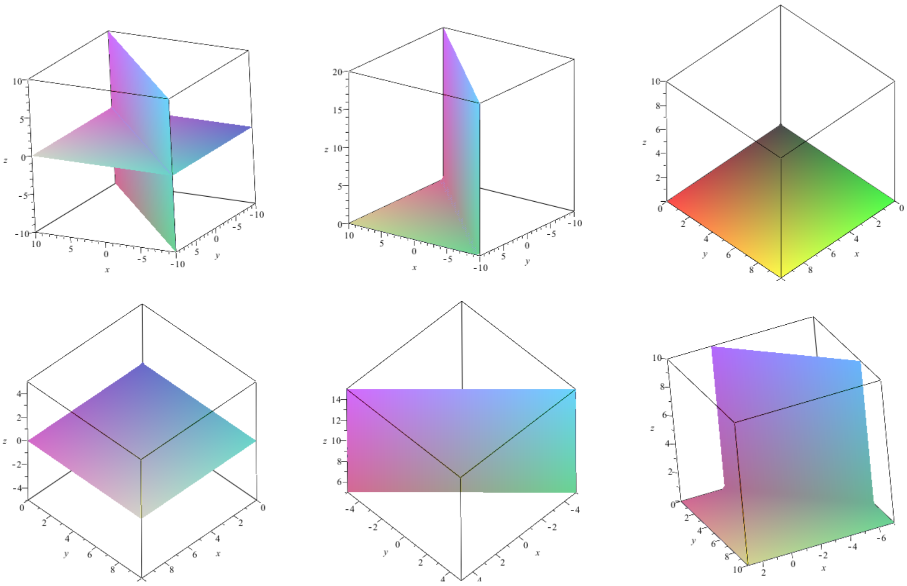
C6(c) $xy + yz + xz + 1 = 0$



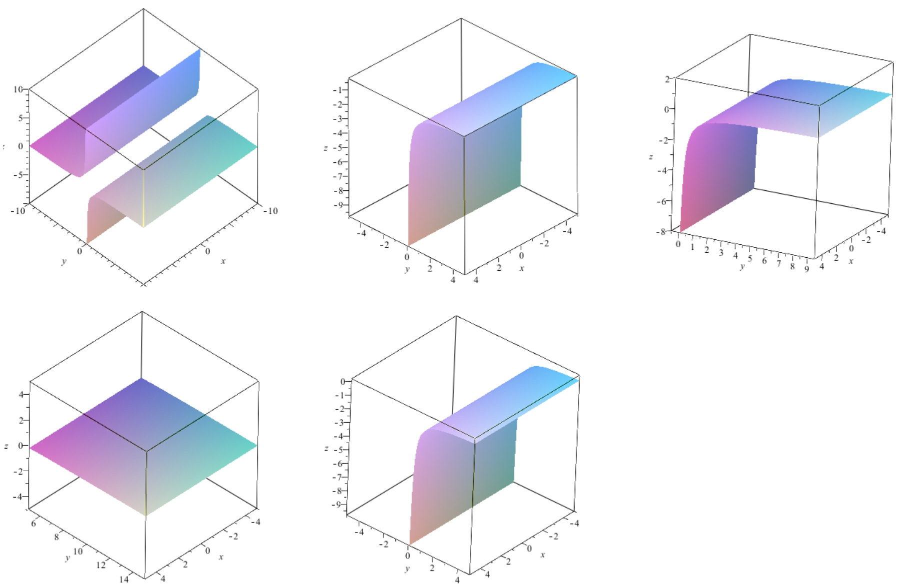
C7 $xz + yz + 1 = 0$



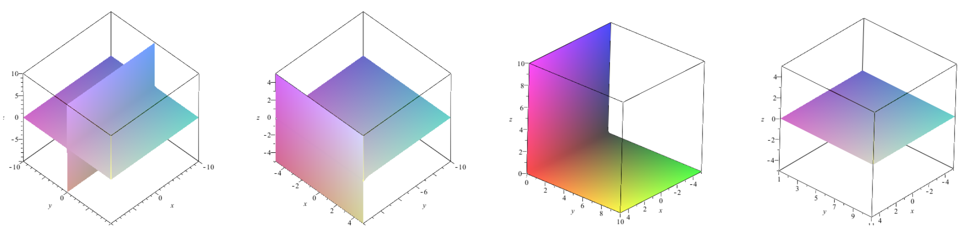
C8 $xz + yz = 0$



C9 $yz + 1 = 0$

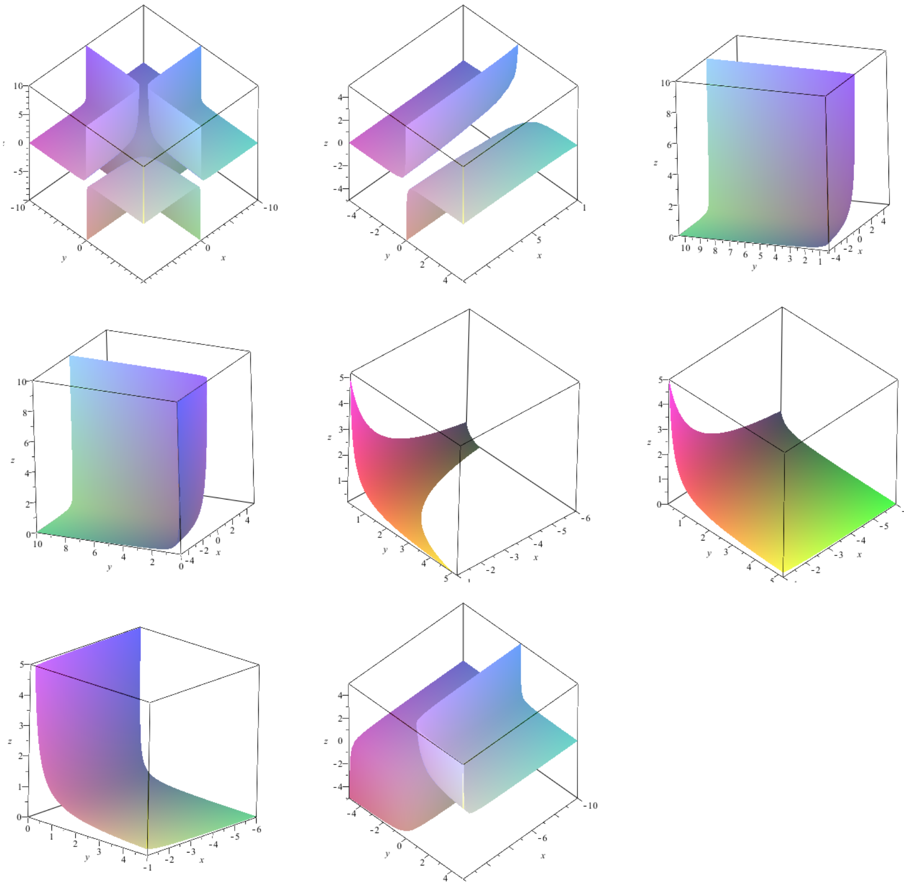


C10 $yz = 0$

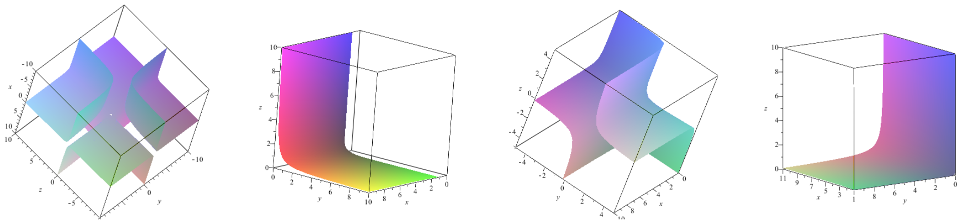


C11 This combination only yields the filled cube.

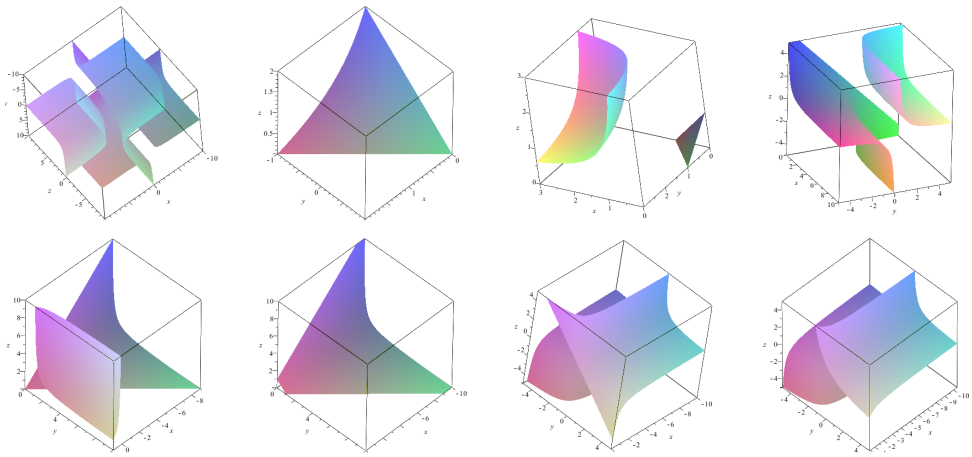
C12 $xyz + 1 = 0$



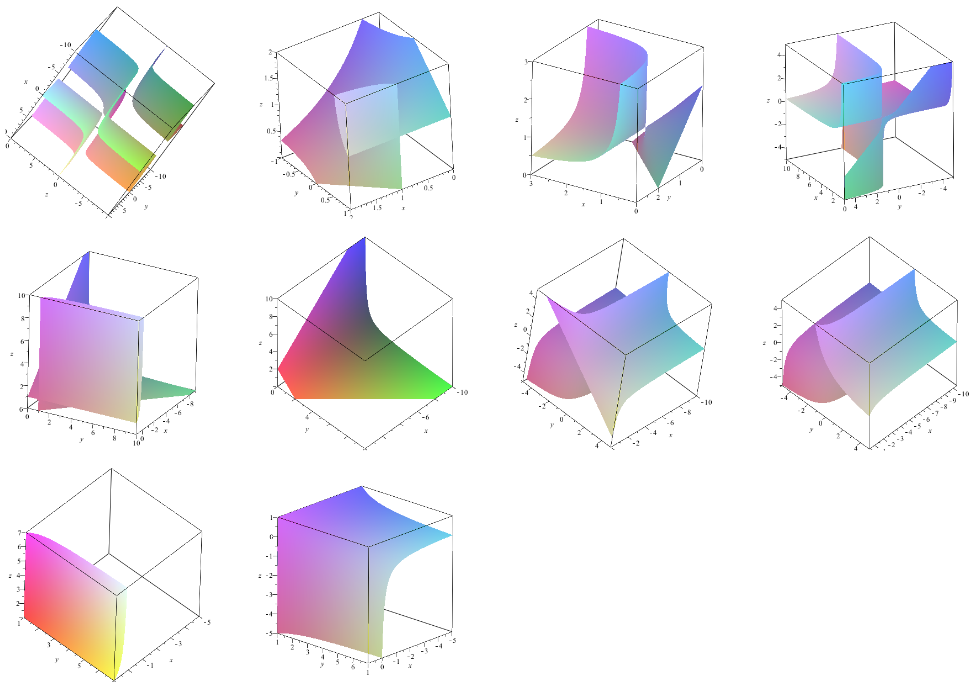
C13 $xyz - x + 1 = 0$



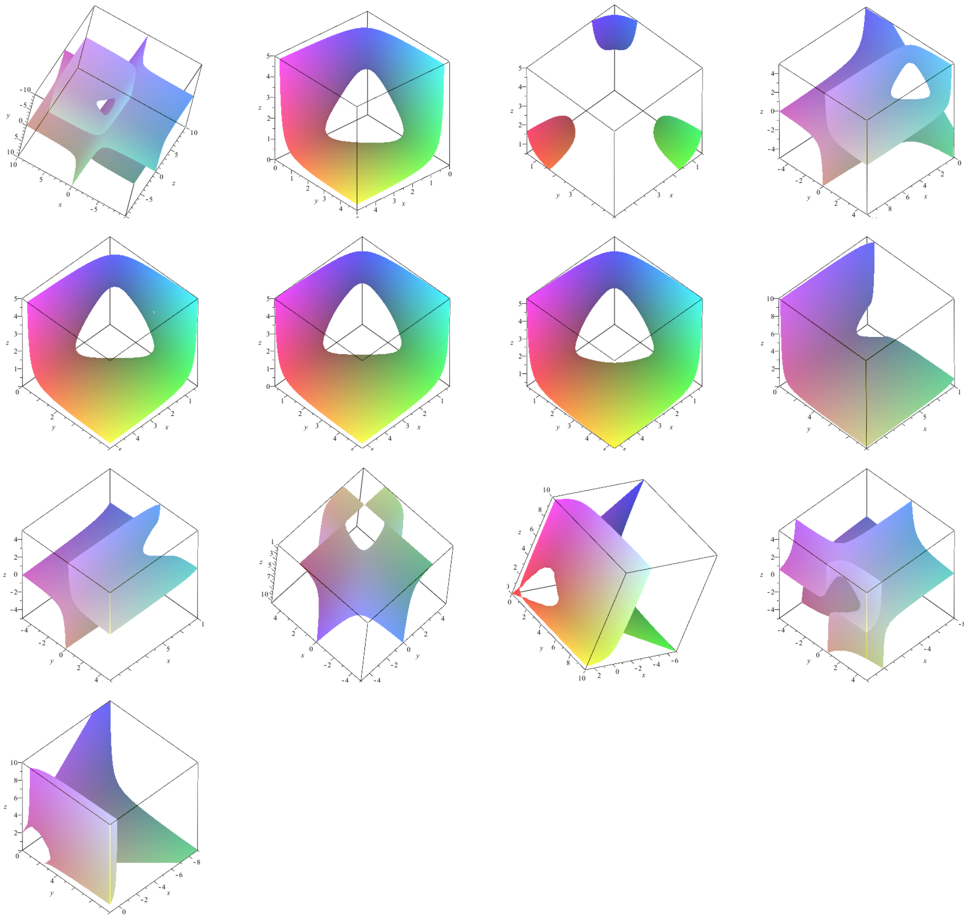
C14(a) $xyz - x - y - z + B_0 = 0,\ |B_0| < 2$



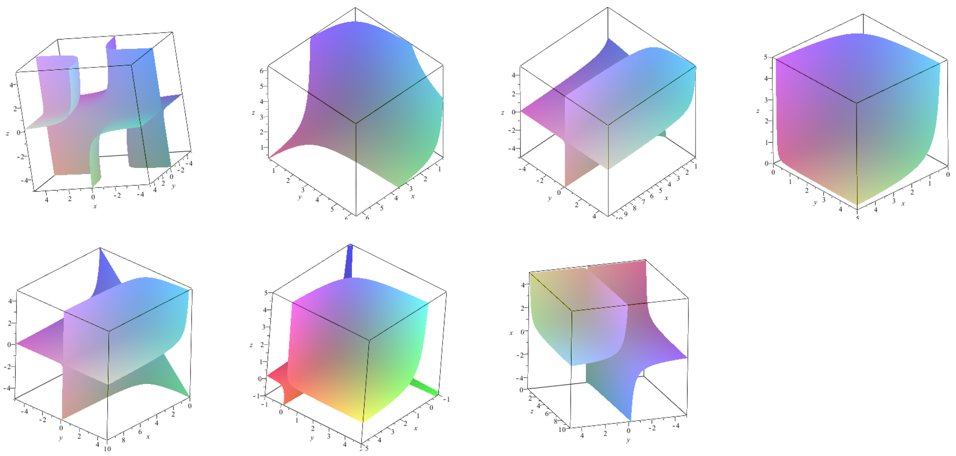
C14(a) $xyz - x - y - z + B_0 = 0,\ |B_0| = 2$



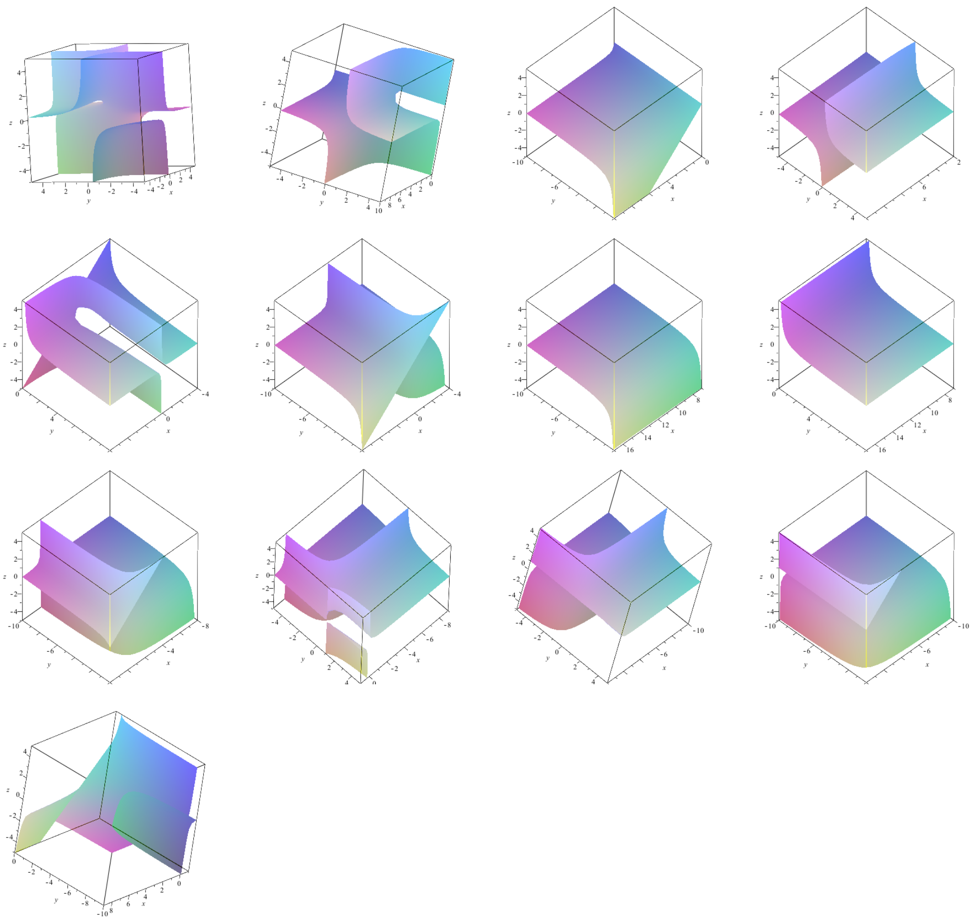
C14(a) $xyz - x - y - z + B_0 = 0,\ |B_0| > 2$



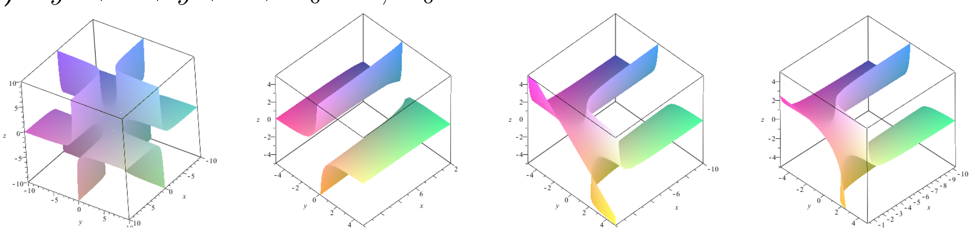
C14(b) $xyz - y - z = 0$



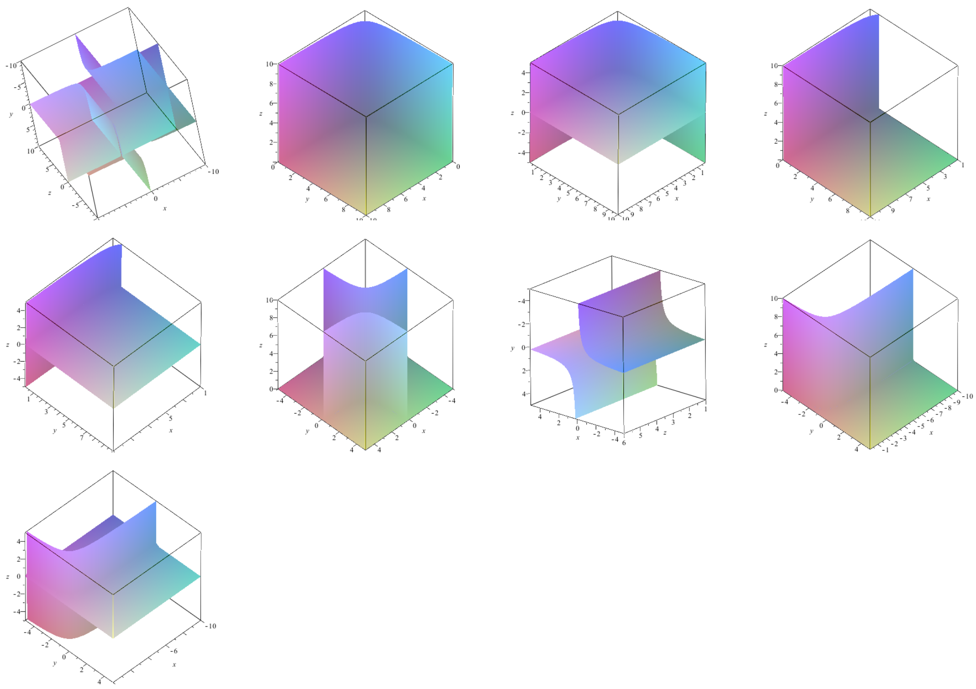
C14(c) $xyz - x - y + 1 = 0$



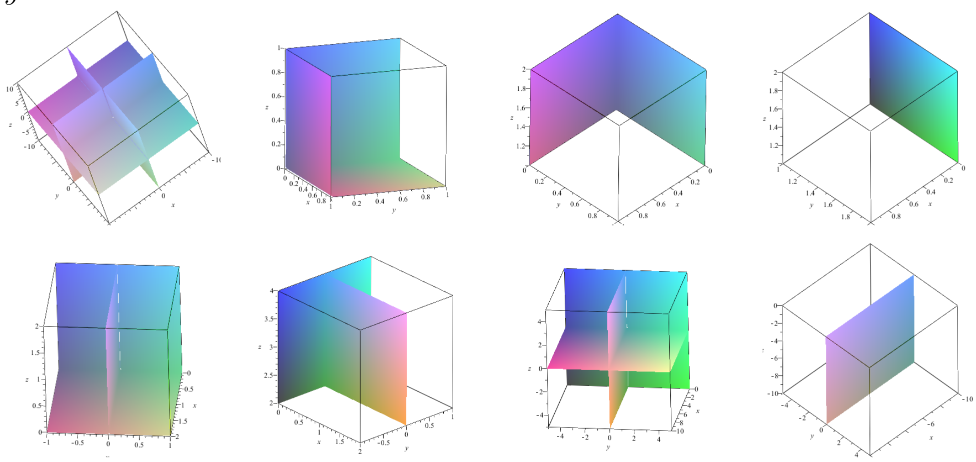
C14(d) $xyz + x + y + z + B_0 = 0,\ B_0 \in \mathbb{R}$



C15 $xyz - z = 0$



C16 $xyz = 0$



CHAPTER 2

Computing the intersections with the cube $[0,1]^3$

The main goal of this chapter is to construct the tiling table for our algorithm. Each triangulation in such a table corresponds to the portion of a solution of the interpolation equation contained in the cube $[0,1]^3$. So, to achieve our purpose, the first step is to investigate which are all the possible truncated surfaces that the members of each class of solutions of the linear interpolation equation are able to produce inside the cube $[0,1]^3$.

2.1 Foliating the portion of surface within the cube

2.1.1 Analyzing the possibilities for each foliation

We begin this section by giving a simple general fact.

Lemma 6. *Let r be a straight line in $\mathbb{R}^3$. Any solution S of the equation*

$$b_0 + b_1x + b_2y + b_3y + b_4xy + b_5xz + b_6yz + b_7xyz = 0 \tag{2.1}$$

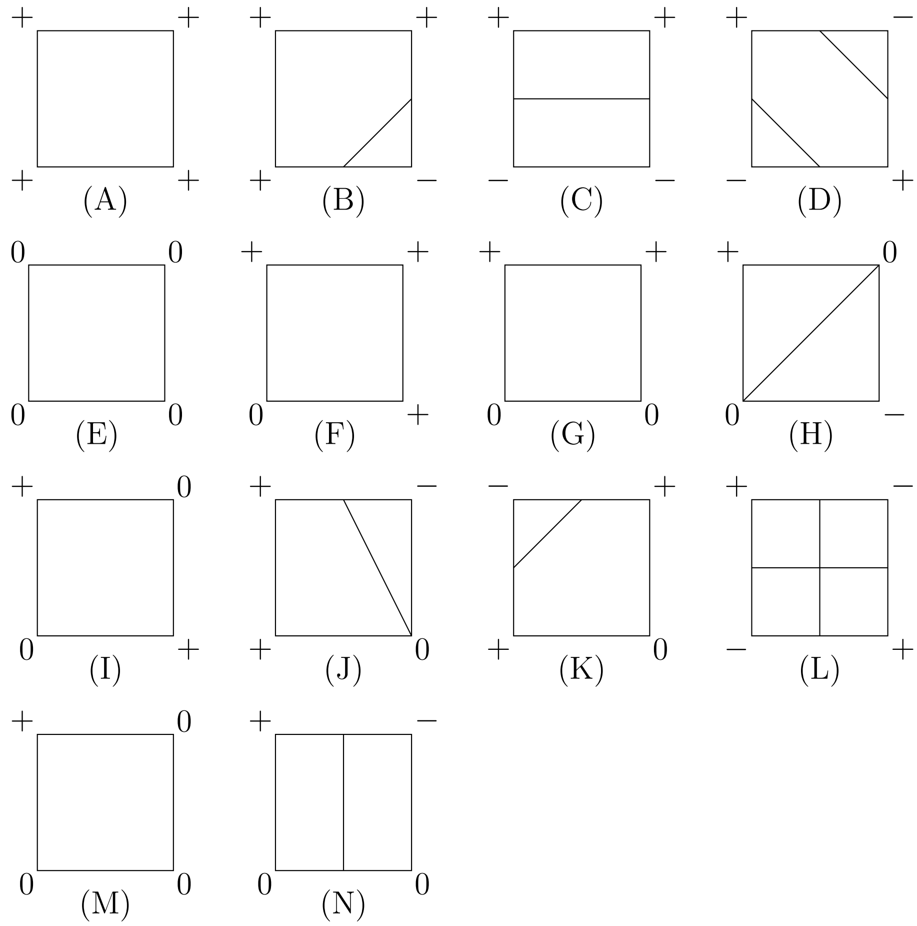
intersects r at most thrice.

Proof. The points on r have the form $(x_0 + at, y_0 + bt, z_0 + ct), t \in \mathbb{R}$, where (x_0, y_0, z_0) is a point on r, and (a, b, c) is a supporting vector of r. Putting $x = x_0+at$, $y = y_0+bt$, and $z = z_0+ct$ in Equation (2.1), we get a polynomial in parameter t, which has at most degree three. Since such a polynomial has at most three roots, we can conclude that S and r have no more than three common points. $\square$

Corollary 4. *Any solution S of Equation (2.1) intersects the edges of the cube $[0,1]^3$ at most once.*

We now focus on classifying all the possible foliations of the portion of S contained in $[0,1]$ by attending to the form of the curve $S \cap \{\theta = x\} \cap [0,1]^3$, for each $x \in [0,1]$. From now on we denote $\{\theta = x\} \cap [0,1]^3$ by τ_x. To our purpose, it suffices to consider the classification of the bi-dimensional case to describe the foliation $\{S \cap \tau_x\}_{x \in [0,1]}$. Notice also that the continuity of S and the previous corollary ensure that the double-orthogonal projection of S onto $[0,1]_x$ (i.e., first project S onto $[0,1]_x \times [0,1]_y$, and then project the result onto $[0,1]_x$) is either a closed subinterval containing one of the ending points of $[0,1]$ or an union of two disjoint closed subintervals, each of them containing one of the ending points of $[0,1]$.

Recall that the classification in $[0,1]^2$ is the following.



(A) (B) (C) (D) (E) (F) (G) (H) (I) (J) (K) (L) (M) (N)

Having disposed of these preliminary steps, we are ready to give the following classification.

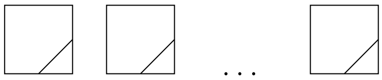
Table 2.1: Non-degenerated truncated foliations

	$P2$	$P3$	$P4$	$P5$	$P6$	$P7$	$P8$	$P9$	$P10$
$I1$	yes	yes	yes	yes	yes	yes	yes	yes	yes
$I2$	yes	yes	yes	yes	yes	yes	yes	yes	yes
$I3$	yes	yes	yes	yes	yes	yes	yes	yes	yes
$I4$	yes	yes	yes	yes	yes	yes	yes	yes	yes
$I5$	yes	yes	yes	yes	yes	yes	yes	yes	yes
$I6$	yes	yes	yes	yes	yes	yes	yes	yes	yes
$I7$	yes	yes	yes	yes	yes	yes	yes	yes	yes
$I8$	yes	yes	yes	yes	yes	yes	yes	yes	yes
$I9$	no	yes	yes	yes	no	yes	no	yes	yes
$I10$	no	yes	yes	yes	no	yes	no	yes	yes
$I11$	no	yes	yes	yes	no	yes	no	yes	yes
$I12$	no	yes	yes	yes	no	yes	no	yes	yes
$I13$	no	yes	yes	yes	no	yes	no	yes	yes
$I14$	no	yes	yes	yes	no	yes	no	yes	yes
$I15$	no	yes	yes	yes	no	yes	no	yes	yes
$I16$	no	yes	yes	yes	no	yes	no	yes	yes
$I17$	no	yes	yes	yes	no	yes	no	yes	yes
$I18$	no	yes	yes	yes	no	yes	no	yes	yes
$I19$	no	yes	yes	yes	no	yes	no	yes	yes
$I20$	no	yes	yes	yes	no	yes	no	yes	yes
$I21$	no	yes	yes	yes	no	yes	no	yes	yes
$I22$	no	yes	yes	yes	no	yes	no	yes	yes
$I23$	no	no	yes	no	no	no	no	no	no
$I33$	no	no	no	no	no	no	no	yes	no
$I34$	no	no	no	no	no	no	no	yes	no
$I35$	no	no	no	no	no	no	no	yes	no
$I36$	no	no	no	no	no	no	no	yes	no
$I37$	no	no	no	no	no	no	no	yes	no
$I38$	no	no	no	no	no	no	no	yes	no
$I39$	no	no	no	no	no	no	no	yes	no
$I40$	no	no	no	no	no	no	no	yes	no
$I42$	no	no	no	no	no	no	no	yes	no
$I43$	no	no	no	no	no	no	no	yes	no
$I44$	no	no	no	no	no	no	no	yes	no
$I45$	no	no	no	no	no	no	no	yes	no

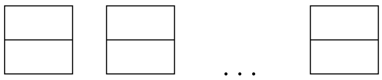
$I46$	no	no	no	no	no	no	no	yes	no

$P2 \bigcap [0,1]^3$ Since $P2$ assures that all the leaves of F_x are straight lines, we have the following truncated foliations:

$I1$ For all $x \in [0,1]$, τ_x matches configuration (B).

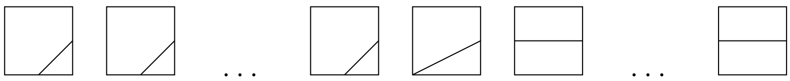


$I2$ For all $x \in [0,1]$, τ_x matches configuration (C).



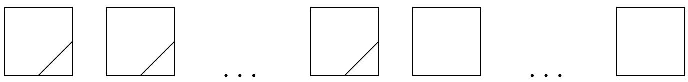
$I3$ There is $0 < a < 1$ such that:

- if $x \in [0,a)$, τ_x matches configuration (B);
- if $x \in (a,1]$, τ_x matches configuration (C);
- If τ_a matches configuration (J).



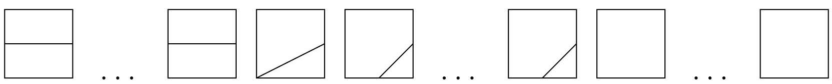
$I4$ There is $0 < a < 1$ such that:

- if $x \in [0,a)$, τ_x matches configuration (B);
- if $x \in (a,1]$, τ_x matches configuration (A);
- τ_a matches configuration (F).



$I5$ There are $0 < a < b < 1$ such that:

- if $x \in [0,a)$, τ_x matches configuration (C);
- if $x \in (a,b)$, τ_x matches configuration (B);
- if $x \in (b,1]$, τ_x matches configuration (A).
- τ_a matches configuration (J);
- τ_b matches configuration (F).



$I6$ There are $0 < a < b < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (B);
- if $x \in (a, b)$, τ_x matches configuration (A);
- if $x \in (b, 1]$, τ_x matches configuration (B);
- τ_a and τ_b match configuration (F).

$I7$ There are $0 < a < b < c < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (B);
- if $x \in (a, b)$, τ_x matches configuration (A);
- if $x \in (b, c)$, τ_x matches configuration (B);
- if $x \in (c, 1]$, τ_x matches configuration (C);
- τ_a and τ_b match configuration (F);
- τ_c matches configuration (J).

$I8$ There are $0 < a < b < c < d < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (C);
- if $x \in (a, b)$, τ_x matches configuration (B);
- if $x \in (b, c)$, τ_x matches configuration (A);
- if $x \in (c, d)$, τ_x matches configuration (B);
- if $x \in (d, 1]$, τ_x matches configuration (C);
- if τ_a and τ_d match configuration (J);
- τ_b and τ_c match configuration (F).

$P3 \bigcap [0,1]^3$ As we have identified a single (connected) arc of hyperbola with a segment of straight line and $P3$ guarantees that all the leaves of F_x are hyperbolas, we get, in addition to the previous, the following truncated foliations:

$I9$ For all $x \in [0, 1]$, τ_x matches configuration (D).

$I10$ There is $0 < a < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (B);
- if $x \in (a, 1]$, τ_x matches configuration (D);
- τ_a matches configuration (K).

$I11$ There are $0 < a < b < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (C);
- if $x \in (a, b)$, τ_x matches configuration (B);

- if $x \in (b, 1]$, τ_x matches configuration (D);
- if τ_a matches configuration (J);
- τ_b matches configuration (K).

$I12$ There is $0 < a < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (D);
- if $x \in (a, 1]$, τ_x matches configuration (A);
- τ_a matches configuration (I).

$I13$ There are $0 < a < b < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (B);
- if $x \in (a, b)$, τ_x matches configuration (D);
- if $x \in (b, 1]$, τ_x matches configuration (A);
- τ_a matches configuration (K);
- τ_b matches configuration (I).

$I14$ There are $0 < a < b < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (B);
- if $x \in (a, b)$, τ_x matches configuration (A);
- if $x \in (b, 1]$, τ_x matches configuration (D);
- τ_a matches configuration (F).
- τ_b matches configuration (I).

$I15$ There are $0 < a < b < c < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (C);
- if $x \in (a, b)$, τ_x matches configuration (B);
- if $x \in (b, c)$, τ_x matches configuration (A);
- if $x \in (c, 1]$, τ_x matches configuration (D);
- τ_a matches configuration (J);
- τ_b matches configuration (F);
- τ_c matches configuration (I).

$I16$ There are $0 < a < b < c < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (B);
- if $x \in (a, b)$, τ_x matches configuration (A);
- if $x \in (b, c)$, τ_x matches configuration (B);
- if $x \in (c, 1]$, τ_x matches configuration (D);
- τ_a and τ_b match configuration (F);
- τ_c matches configuration (K).

$I17$ There are $0 < a < b < c < d < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (C);
- if $x \in (a, b)$, τ_x matches configuration (B);
- if $x \in (b, c)$, τ_x matches configuration (A);
- if $x \in (c, d)$, tau_x matches configuration (B);
- if $x \in (d, 1]$, tau_x matches configuration (D);
- τ_a matches configuration (J);
- τ_b and τ_c match configuration (F);
- τ_d matches configuration (K).

$I18$ There are $0 < a < b < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (D);
- if $x \in (a, b)$, τ_x matches configuration (A);
- if $x \in (b, 1]$, τ_x matches configuration (D);
- τ_a and τ_b match configuration (F).

$I19$ There are $0 < a < b < c < d < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (B);
- if $x \in (a, b)$, τ_x matches configuration (D);
- if $x \in (b, c)$, τ_x matches configuration (A);
- if $x \in (c, d)$, τ_x matches configuration (B);
- if $x \in (d, 1]$, τ_x matches configuration (D);
- τ_a and τ_d match configuration (K);
- τ_b matches configuration (I);
- τ_c matches configuration (F).

$I20$ There are $0 < a < b < c < d < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (C);
- if $x \in (a, b)$, τ_x matches configuration (B);
- if $x \in (b, c)$, τ_x matches configuration (D);
- if $x \in (c, d)$, τ_x matches configuration (A);
- if $x \in (d, 1]$, τ_x matches configuration (B);
- τ_a matches configuration (J);
- τ_b matches configuration (K);
- τ_c matches configuration (I);
- τ_d matches configuration (F).

$I21$ There are $0 < a < b < c < d < e < 1$ such that:

- if $x \in [0, a)$, τ_x matches configuration (C);
- if $x \in (a, b)$, τ_x matches configuration (B);
- if $x \in (b, c)$, τ_x matches configuration (D);
- if $x \in (c, d)$, τ_x matches configuration (A);
- if $x \in (d, e)$, τ_x matches configuration (B);
- if $x \in (e, 1]$, τ_x matches configuration (D);
- τ_a matches configuration (J);
- τ_b and τ_e match configuration (K);
- τ_c matches configuration (I);
- τ_d matches configuration (F).

$P4 \bigcap [0,1]^3$ This possibility asserts that all the leaves of F_x are planes. So, we only have a possible truncated foliation:

$I22$ For all $x \in [0, 1]$, τ_x has configuration (E).

$P5 \bigcap [0,1]^3$ This possibility states that all the leaves of F_x are empty, except for one of them which is a plane. Hence, the only two new truncated foliation that this possibility offers are:

$I23$ For all $x \in [0, 1]$, $x \neq a$, τ_x presents configuration (A), and τ_a has configuration (E).

$I24$ For all $x \in [0, 1]$, τ_x has configuration (G).

$P6 \bigcap [0,1]^3$ This possibility states that all the leaves of F_x are straight lines, except for one of them which is empty. Since the surface has an asymptotic behavior respect this leave, this possibility contributes with no truncated foliations besides those obtained from $P2$.

$P7 \bigcap [0,1]^3$ This possibility declares that all the leaves of F_x are hyperbolas, except for one of them which is empty. Due to the fact that the surface behaves asymptotically in relation to this leave, this possibility contributes with no truncated foliations beside those obtained from $P3$.

$P8 \bigcap [0,1]^3$ In the case of this possibility, all the leaves of F_x are straight lines, excepting by one of then which is a plane. So, apart from the truncated foliations obtained from $P2$, this possibility only contributes with the following **degenerated** cases:

$I25$ For all $x \in [0, 1]$, $x \neq a$, τ_x has configuration (B). τ_a presents configuration (E).

$I26$ For all $x \in [0,1]$, $x \neq a$, τ_x presents configuration (C). τ_a has configuration (E).

$I27$ There is $0 < b < 1$, $b \neq a \in (0,1)$ such that:

- if $x \in [0,b)$, τ_x matches configuration (C);
- if $x \in (b,1]$, τ_x matches configuration (B);
- τ_a matches configuration (E);
- τ_b matches configuration (J).

$I28$ There are $0 < a < b < 1$ such that:

- if $x \in [0,b)$, τ_x matches configuration (B);
- if $x \in (b,1)$, τ_x matches configuration (A);
- τ_a matches configuration (E);
- τ_b matches configuration (F).

$I29$ There are $0 < b < a < c < 1$ such that:

- if $x \in [0,b)$, τ_x matches configuration (C);
- if $x \in (b,c)$, τ_x matches configuration (B);
- if $x \in (c,1]$, τ_x matches configuration (A);
- τ_a matches configuration (E);
- τ_b matches configuration (J);
- τ_c matches configuration (F).

$I30$ There are $0 < a < b < c < 1$ such that:

- if $x \in [0,b)$, τ_x matches configuration (B);
- if $x \in (b,c)$, τ_x matches configuration (A);
- if $x \in (c,1]$, τ_x matches configuration (B);
- τ_a matches configuration (E);
- τ_b and τ_c match configuration (F).

$I31$ There are $0 < b < c < d < 1$ and $a \in (0,c)\bigcup(d,1)$, $a \neq b$, such that:

- if $x \in [0,b)$, τ_x matches configuration (C);
- if $x \in (b,c)$, τ_x matches configuration (B);
- if $x \in (c,d)$, τ_x matches configuration (A);
- if $x \in (d,1]$, τ_x matches configuration (B);
- τ_a matches configuration (E);
- τ_b matches configuration (J);
- τ_c and τ_d match configuration (F).

$I32$ There are $0 < b < c < d < e < 1$ and $a \in (0, c) \bigcup (d, 1)$, $a \neq b, e$, such that:

- if $x \in [0, b)$, τ_x matches configuration (C);
- if $x \in (b, c)$, τ_x matches configuration (B);
- if $x \in (c, d)$, τ_x matches configuration (A);
- if $x \in (d, e)$, τ_x matches configuration (B);
- if $x \in (e, 1]$, τ_x matches configuration (C);
- τ_a matches configuration (E);
- τ_b and τ_e match configuration (J);
- τ_c and τ_d match configuration (F).

$P9 \bigcap [0, 1]^3$ In this possibility, all the leaves of F_x are hyperbolas, excepting by one of them which is a straight line. In view of this, in addition to the truncated foliations provided by $P3$, we have the following:

$I33$ For all $x \in [0, 1]$, $x \neq a$, τ_x matches configuration (D). τ_a matches configuration (H).

$I34$ There are $0 < b < a \leq 1$ such that:

- if $x \in [0, b)$, τ_x matches configuration (B);
- if $x \in (b, 1]$, $x \neq a$, τ_x matches configuration (D);
- τ_a matches configuration (H);
- τ_b matches configuration (K).

$I35$ There are $0 < b < c < a \leq 1$ such that:

- if $x \in [0, b)$, τ_x matches configuration (C);
- if $x \in (b, c)$, τ_x matches configuration (B);
- if $x \in (c, 1]$, $x \neq a$, τ_x matches configuration (D)
- τ_a matches configuration (H);
- τ_b matches configuration (J);
- τ_c matches configuration (K).

$I36$ There are $0 < a < b < 1$ such that:

- if $x \in [0, b)$, $x \neq a$, τ_x matches configuration (D);
- if $x \in (b, 1]$, τ_x matches configuration (A);
- τ_a matches configuration (H);
- τ_b matches configuration (I).

$I37$ There are $0 < b < a < c < 1$ such that:

- if $x \in [0, b)$, τ_x matches configuration (B);
- if $x \in (b, c)$, $x \neq a$, τ_x matches configuration (D);
- if $x \in (c, 1]$, τ_x matches configuration (A);
- τ_a matches configuration (H);
- τ_b matches configuration (K);
- τ_c matches configuration (I).

$I38$ There are $0 < b < c < a < d < 1$ such that:

- if $x \in [0, b)$, τ_x matches configuration (C);
- if $x \in (b, c)$, τ_x matches configuration (B);
- if $x \in (c, d)$, $x \neq a$, τ_x matches configuration (D);
- if $x \in (d, 1]$, τ_x matches configuration (A);
- τ_a matches configuration (H);
- τ_b matches configuration (J);
- τ_c matches configuration (K);
- τ_d matches configuration (I).

$I39$ There are $0 < b < c < a \leq 1$ such that:

- if $x \in [0, b)$, τ_x matches configuration (B);
- if $x \in (b, c)$, τ_x matches configuration (A);
- if $x \in (c, 1]$, $x \neq a$, τ_x matches configuration (D);
- τ_a matches configuration (H);
- τ_b matches configuration (F);
- τ_c matches configuration (I).

$I40$ There are $0 < b < c < d < a \leq 1$ such that:

- if $x \in [0, b)$, τ_x matches configuration (C);
- if $x \in (b, c)$, τ_x matches configuration (B);
- if $x \in (c, d)$, τ_x matches configuration (A);
- if $x \in (d, 1]$, $x \neq a$ τ_x matches configuration (D);
- τ_a matches configuration (H);
- τ_b matches configuration (J);
- τ_c matches configuration (F);
- τ_d matches configuration (I).

$I41$ There are $0 < b < c < d < a \leq 1$ such that:

- if $x \in [0, b)$, τ_x matches configuration (B);
- if $x \in (b, c)$, τ_x matches configuration (A);

- if $x \in (c, d)$, τ_x matches configuration (B);
- if $x \in (d, 1]$, $x \neq a$, τ_x matches configuration (D);
- τ_a matches configuration (H);
- τ_b and τ_c match configuration (F);
- τ_d matches configuration (K).

$I42$ There are $0 < b < c < d < e < a \leq 1$ such that:

- if $x \in [0, b)$, τ_x matches configuration (C);
- if $x \in (b, c)$, τ_x matches configuration (B);
- if $x \in (c, d)$, τ_x matches configuration (A);
- if $x \in (d, e)$, τ_x matches configuration (B);
- if $x \in (e, 1]$, $x \neq a$, τ_x matches configuration (D);
- τ_a matches configuration (H);
- τ_b matches configuration (J);
- τ_c and τ_d match configuration (F);
- τ_e matches configuration (K).

$I43$ There are $0 < a < b < c < 1$ such that:

- if $x \in [0, b]$, $x \neq a$, τ_x matches configuration (D);
- if $x \in (b, c)$, τ_x matches configuration (A);
- if $x \in (c, 1]$, τ_x matches configuration (D);
- τ_a matches configuration (H);
- τ_b and τ_c match configuration (I).

$I44$ There are $0 < b < c < d < e < 1$, $a \in (b, c) \bigcup (e, 1]$ such that:

- if $x \in [0, b)$, τ_x matches configuration (B);
- if $x \in (b, c)$, τ_x matches configuration (D);
- if $x \in (c, d)$, τ_x matches configuration (A);
- if $x \in (d, e)$, τ_x matches configuration (B);
- if $x \in (e, 1]$, $x \neq a$, τ_x matches configuration (D);
- τ_a matches configuration (H);
- τ_b and τ_e matches configuration (K);
- τ_c matches configuration (I);
- τ_a matches configuration (F).

$I45$ There are $0 < b < c < d < e < f < 1$, $a \in (c, d) \bigcup (f, 1]$ such that:

- if $x \in [0, b)$, τ_x matches configuration (C);
- if $x \in (b, c)$, τ_x matches configuration (B);

- if $x \in (c, d)$, $x \neq a$, τ_x matches configuration (D);
- if $x \in (d, e)$, τ_x matches configuration (A);
- if $x \in (e, f)$, τ_x matches configuration (B);
- if $x \in (f, 1]$, $x \neq a$, τ_x matches configuration (D);
- τ_a matches configuration (H);
- τ_b matches configuration (J);
- τ_c and τ_f match configuration (K);
- τ_d matches configuration (I);
- τ_e matches configuration (F).

$I46$ There are $0 < b < c < d < e < f < g < 1$, $a \in (c, d) \bigcup (e, f)$ such that:

- if $x \in [0, b)$, τ_x matches configuration (C);
- if $x \in (b, c)$, τ_x matches configuration (B);
- if $x \in (c, d)$, τ_x matches configuration (D);
- if $x \in (d, e)$, τ_x matches configuration (A);
- if $x \in (e, f)$, τ_x matches configuration (D);
- if $x \in (f, g)$, τ_x matches configuration (B);
- if $x \in (g, 1]$, τ_x matches configuration (C);
- τ_a matches configuration (H);
- τ_b and τ_g match configuration (J);
- τ_c and τ_f match configuration (K);
- τ_d and τ_e match configuration (F).

$P10 \bigcap [0, 1]^3$ This possibility asserts that all the leaves of F_x are hyperbolas, excepting by one of them which is a plane. Thus, beside the truncated foliations provided by $P3$, we obtain the various **degenerated** ones. Replacing τ_a *matches configuration* (B) by τ_a *matches configuration* (E) in each the statement of each of the truncated foliation presented in the analysis of $P9$, we get the firsts fourteen of $P10$. In addition, we have:

$I61$ For all $x \in [0, 1]$, τ_x matches configuration (L).

$I62$ For all $x \in [0, 1]$, τ_x matches configuration (M).

$I63$ For all $x \in [0, 1]$, τ_x matches configuration (N).

2.1.2 Analyzing the combinations of foliation

The purpose of this section is to illustrates how obtain all the possible combinations (I_i, I_j, I_k) of truncated foliations. To achieve this, we shall rely on the results previously obtained for combinations of entire foliations. In our analysis we will assume that the coordinate "x" varies from front to back, the "y" varies from left to right, and the "z" varies from bottom to top.

$(P2, P2, P2)$ In this case, the general equation becomes $b_0 + b_1x + b_2y + b_3z = 0$, $b_1, b_2, b_3 \neq 0$, and hence any solution of this equation is a plane.

Let us begin by assuming that $I1$ appears in the combination; for instance, consider that the truncation of F_x leads to $I1$. This means the for all value of $x \in [0, 1]$, the surface intersects exactly a vertical and an horizontal edges of the square τ_x. To simplify the analysis, suppose that the surface intersects the bottom and the right edges of τ_x. (To fix ideas, we always assume that along this section.) Thus, when we analyze the truncation of F_y, two situations may occur: (a) the surface intersects τ_0 ($y = 0$!) at a bottom vertex, and (b) the surface does not intersect τ_0. In the case of (a), the truncation of F_y either is $I1$ or $I3$ depending on whether the surface contains the diagonal of the bottom face of the cube or not. If the truncation of F_y is $I1$, then the truncation of F_z either is $I1$ or $I4$, depending on whether the surface contains the point (0,1,1) or not; but if the truncation of F_y is $I3$, then using a similar argument we can conclude that the truncation of F_z either is $I3$ or $I5$. We thus get the combinations $(I1, I1, I1)$, $(I1, I1, I4)$, $(I1, I3, I3)$, and $(I1, I3, I5)$. Now, in the case of (b), the truncation of F_y either is $I4$ or $I5$, depending on whether the surface contains the point $(1, 1, 0)$ or not. If the truncation of F_y is $I4$, then the truncation of F_z either is $I1$ or $I4$ according to whether the point (0,1,1) lies in the surface or not. Now, if F_y is $I5$, then in view of the same statement we can deduce that the truncation of F_z either is $I3$ or $I5$. Thus, we obtain three new combinations: $(I1, I3, I4)$, $(I1, I4, I4)$, and $(I1, I5, I5)$.

Suppose now that the truncation of F_x is $I2$. In this case it is not difficult to see that both the truncation of F_y and the truncation of F_z are $I2$. This contributes with a single combination, namely $(I2, I2, I2)$.

Consider that the truncation of F_x is $I3$. There is no restriction of generality in assuming that in the squares τ_x with configuration (B), the surface intersects the bottom and the right edges; and in the squares τ_x with configuration (C), the surface intersects the vertical edges. It is not difficult to see that the truncation of F_y uniquely admits the possibilities $I1$ and $I3$ according to whether the point $(0,1,0)$ belongs to the surface or not. If the truncation of F_y is $I1$, then the possibilities for F_z are $I3$ and $I5$ and hence no new combinations are obtained. If the truncation of F_y is $I3$, then depending on whether the point $(1,1,1)$ is in the surface or not, we get that the truncation of F_z is $I3$ and $I5$, respectively. We thus get two new combinations, namely $(I3, I3, I3)$ and $(I3, I3, I5)$.

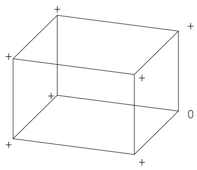
Consider now that the truncation of F_x is $I4$. There is no loss of generality in assuming that the squares τ_x with combination (B) are intersected by the surface at the bottom a right edges. Thus, the family of squares τ_x, $x \in [0,1]$ is such that the firsts squares are intersected at the bottom and right edges, then appears a square that is intersected by the surface at a unique point, namely the point corresponding to $y = 1$ and $z = 0$, and the remaining squares have intersection empty with the surface. From this, it is not hard to see that the truncation of F_y to be either $I1$ or $I4$. Doing a similar reasoning, we can deduce in both cases the truncation of F_z either is $I1$ or $I4$. Thus, the only new combination we obtain this time is: $(I4, I4, I4)$.

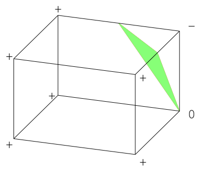
Let the truncation of F_x be $I5$. Assume, without loss of generality, that in the squares τ_x with configuration (B), the surface intersects the bottom and the right edges; and in the squares τ_x with configuration (C), the surface intersects the vertical edges. In this way we obtain that the family of squares τ_x, $x \in [0,1]$, is such that the firsts squares are intersected by the surface at the vertical edges, these are followed by squares intersected by the surfaces at the bottom and right edges, then we find a square uniquely intersected at the point corresponding to $y = 1$ and $z = 0$, and the final squares have no intersection with the surface. Thus, we can verify without much of effort that the truncation F_y can only be $I1$. This implies that the truncation of F_z either is $I3$ or $I5$, depending on whether the surface contains the point $(0,1,1)$ or not. This gives no new combinations.

Finally, using the fact that the solutions of the equation $b_0 + b_1x + b_2y + b_3z = 0$, $b_1, b_2, b_3 \neq 0$, is plane, we can deduce without much of effort that there is no combination containing any of the truncations $I6 - I8$.

2.2 The configurations of the cube

In this section we list all the possible cases with null vertices and their respective solutions. It is known from [11] that the number of cases when null vertices are allowed is 147. Here we ratify this result. The cases without null vertices are fifteen and were exhibited in [3]. To avoid unnecesary repetions, we omit them in our presentation, however we begin the enumeration of our cases in 15 to indicate that those are the firts of the list.


Case 15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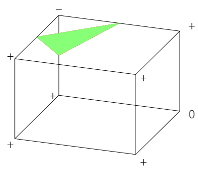
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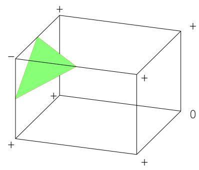
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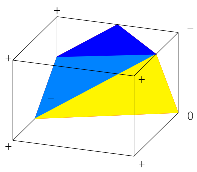
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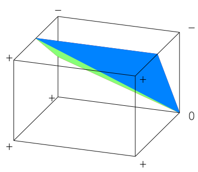
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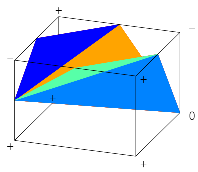
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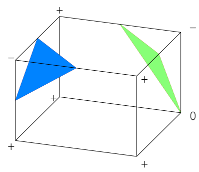
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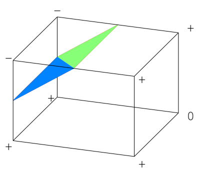
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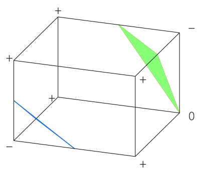
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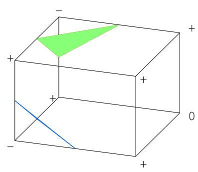
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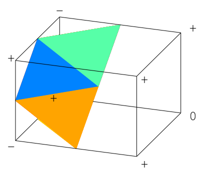
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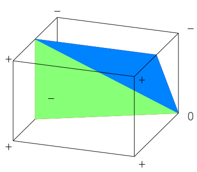
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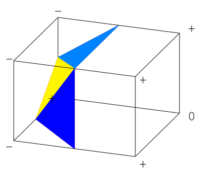
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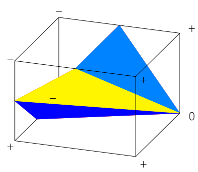
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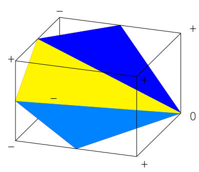
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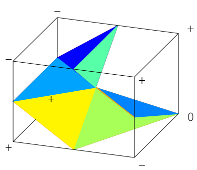
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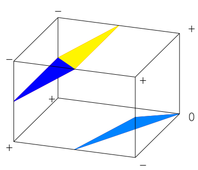
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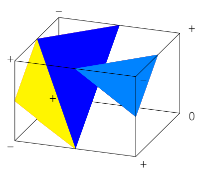
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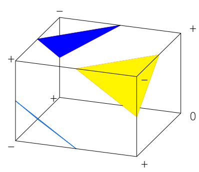
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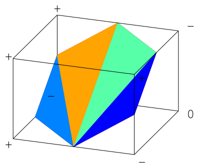
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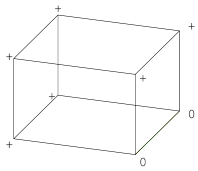
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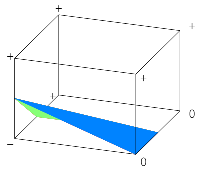
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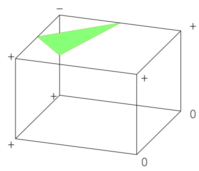
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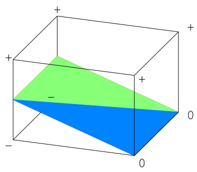
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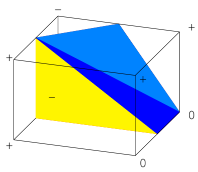
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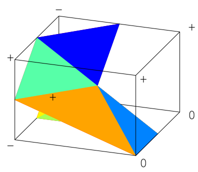
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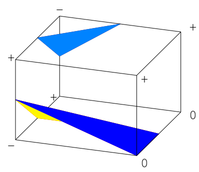
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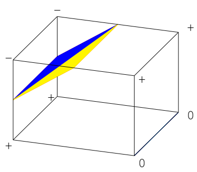
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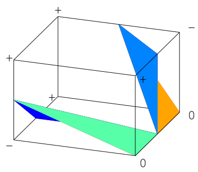
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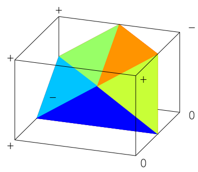
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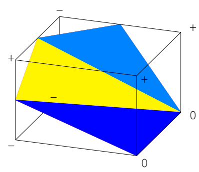
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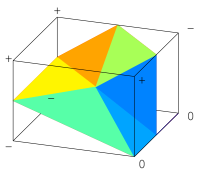
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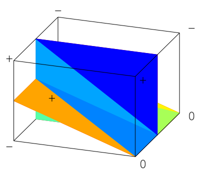
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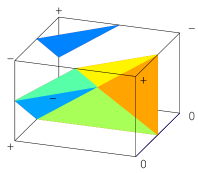
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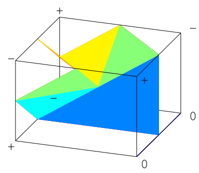
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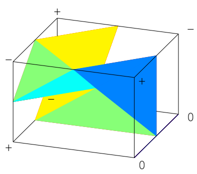
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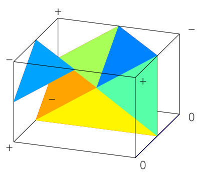
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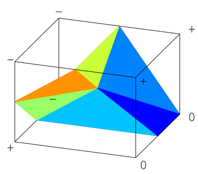
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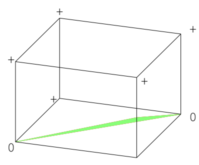
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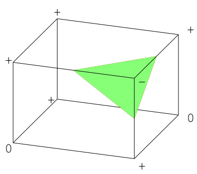
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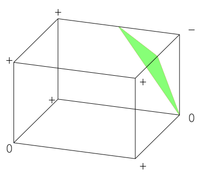
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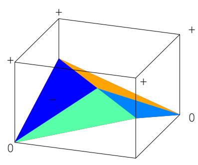
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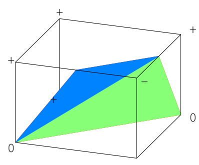
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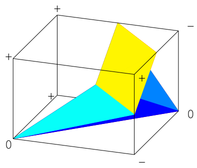
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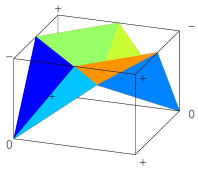
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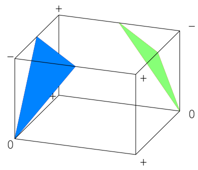
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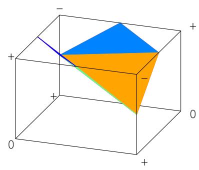
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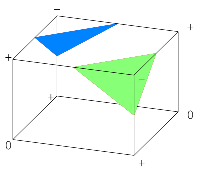
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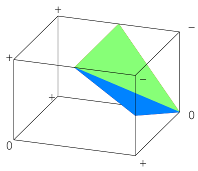
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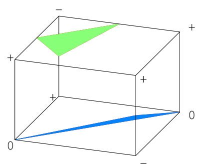
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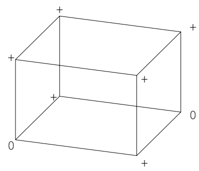
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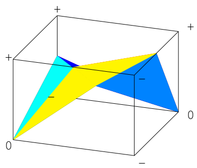
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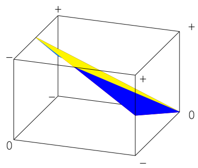
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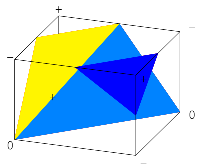
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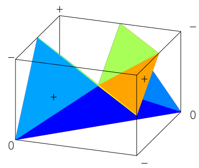
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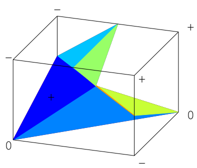
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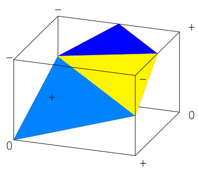
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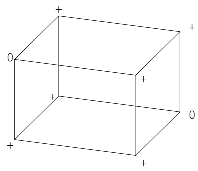
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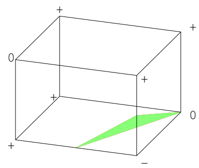
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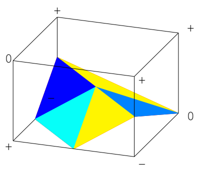
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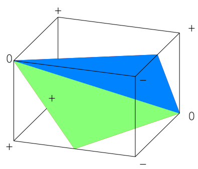
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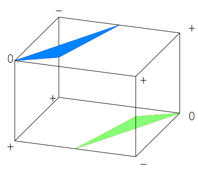
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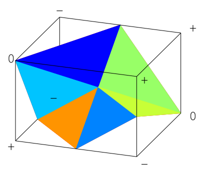
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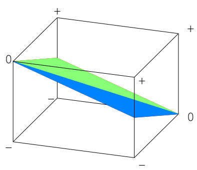
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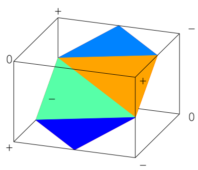
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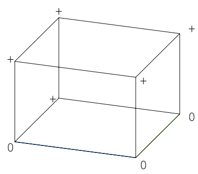
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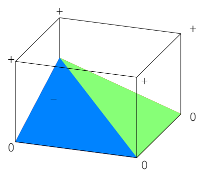
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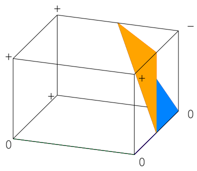
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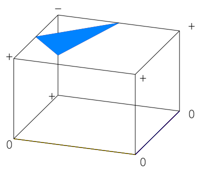
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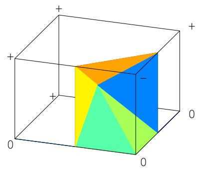
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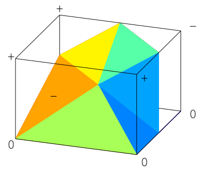
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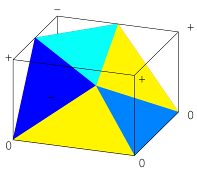
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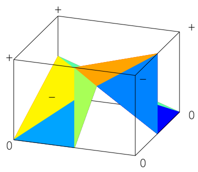
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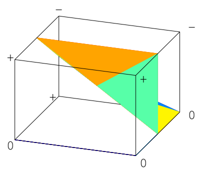
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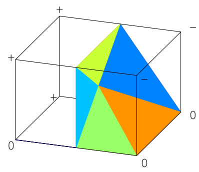
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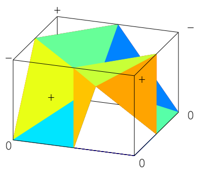
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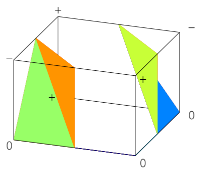
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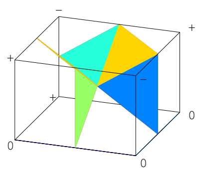
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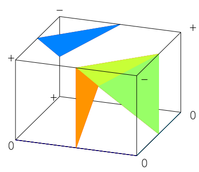
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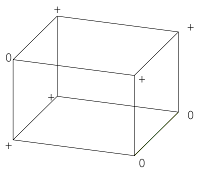
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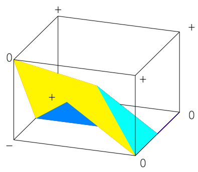
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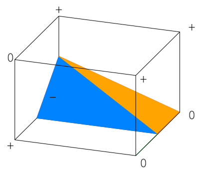
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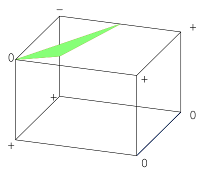
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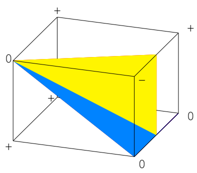
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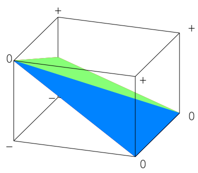
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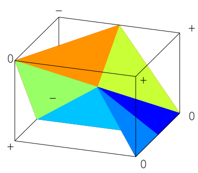
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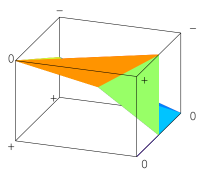
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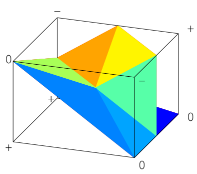
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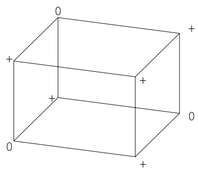
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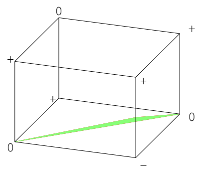
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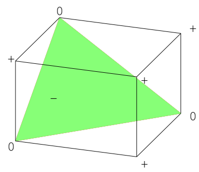
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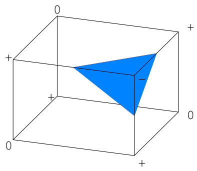
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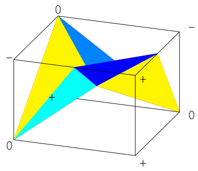
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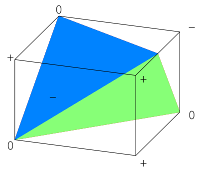
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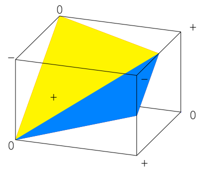
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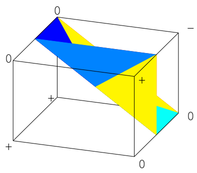
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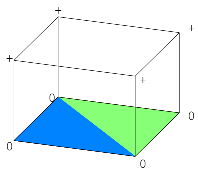
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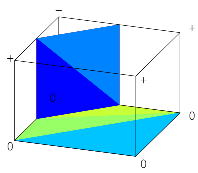
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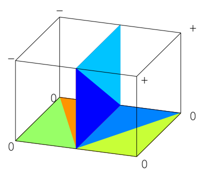
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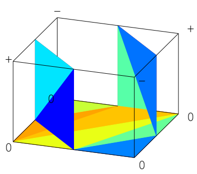
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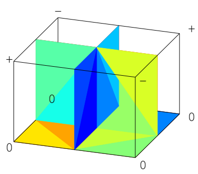
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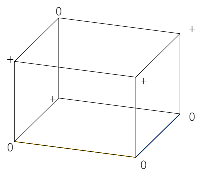
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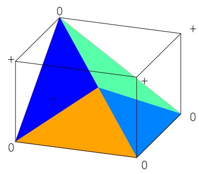
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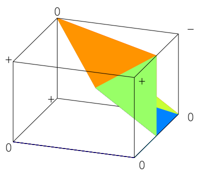
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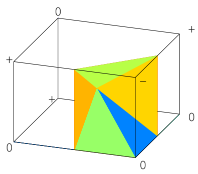
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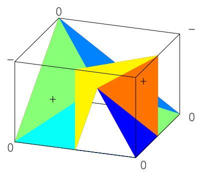
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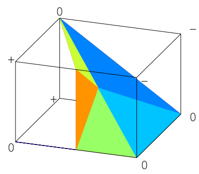
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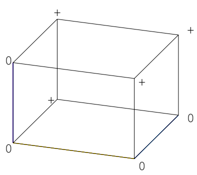
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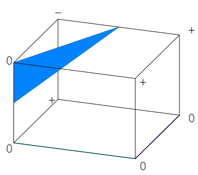
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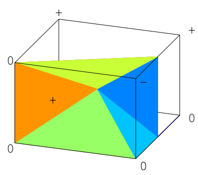
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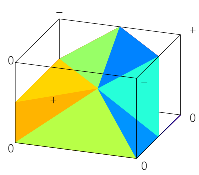
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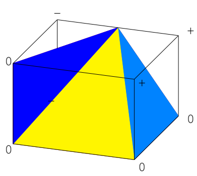
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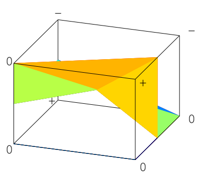
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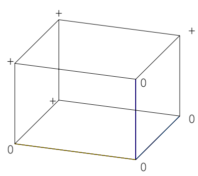
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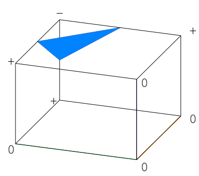
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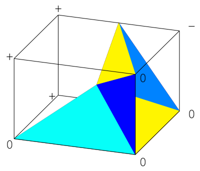
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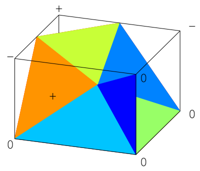
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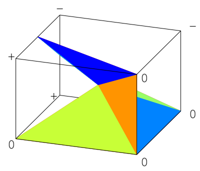
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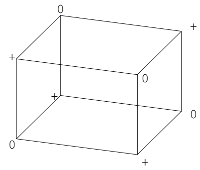
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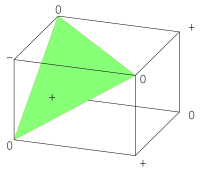
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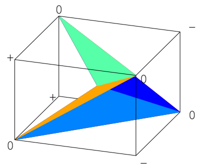
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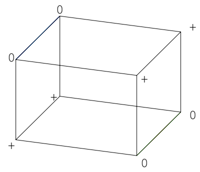
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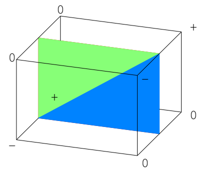
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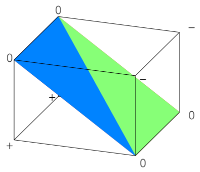
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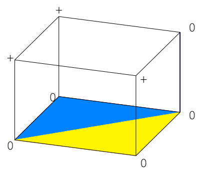
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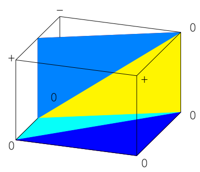
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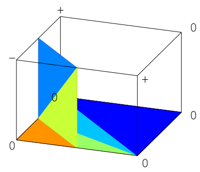
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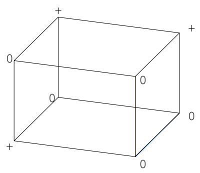
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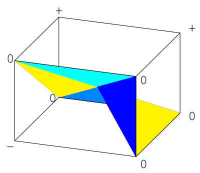
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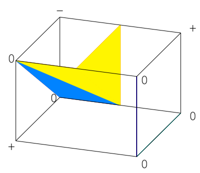
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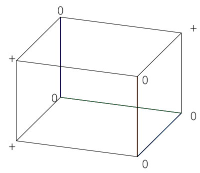
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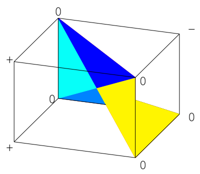
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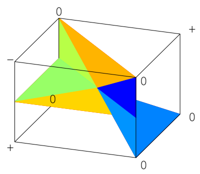
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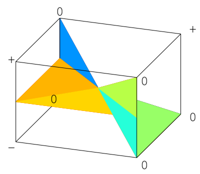
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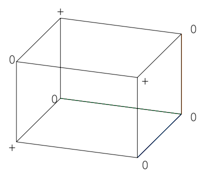
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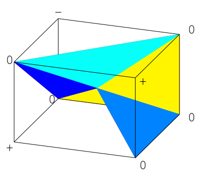
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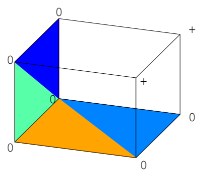
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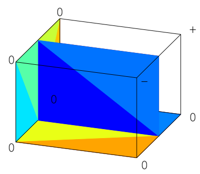
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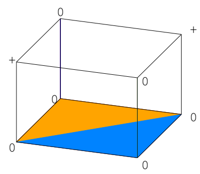
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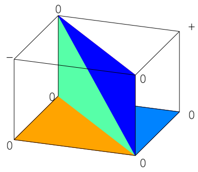
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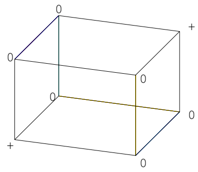
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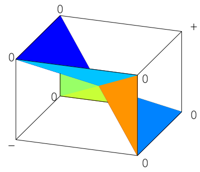
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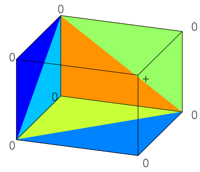
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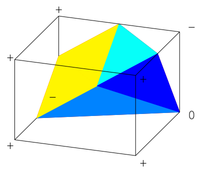
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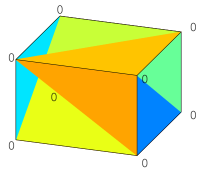

Case 146

2.3 The Marching Cubes algorithm: an advancement

In this section, we introduce an improvement of the algorithm proposed by Nielson in [2], which in turn constitutes an improvement of the original Marching Cubes method. The implementation of our algorithm is based on that proposed by Lewiner et al. [3].

2.3.1 General description of Marching Cubes method

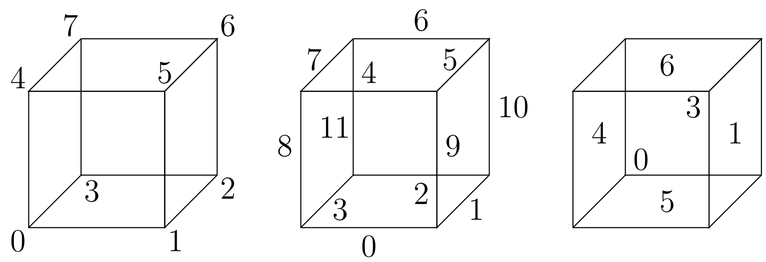
The Marching Cubes method yields a triangle mesh of the preimage $f^{-1}(c)$ of the real value c by the scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ which is sampled over a cuberille grid. The method processes one cube at a time, tiling in each case the portion of the surface contained in the current cube. Each vertex v has either assigned a positive or negative sign accordingly to the sign of $f(v) - c$, which leads to $2^8 = 256$ possible configurations of a cube.

The purpose of the algorithm is to produce a surface homeomorphic to $F^{-1}(c)$, where F matches with f at the vertices of the cuberille, and is trilinear within each cube of the grid. To avoid cracks, the method performs topological tests on ambiguous faces of a cube. The same test should be done on the contiguous cube in order to attain a coherent transition from a cube to the other.

The usual test to solve face ambiguities was introduced by Nielson et al. [4] and consists of verifying which of the two pairs of diagonally-opposed vertices of the cube has the same sign as the middle point of the cube. The resolution of face ambiguities assures the no presence of cracks. However, to accomplish an appropriate solution in the afore-mentioned sense, we also need to solve internal ambiguities. In this regard, Lewiner et al. introduced a plausible technique based on an extended lookup table and an enhanced analysis of each cube [3].

2.3.2 Our algorithm

As we mentioned before, our algorithm is an improvement of the version presented by Lewiner et al. in [3]. This algorithm is mainly based on three tables: the configuration table, the test table, and the tiling table. The configuration table has 256 entries, each one of them represented as an 8-bit word in the following way: the ith bit is equal to 1 (resp. 0) if $f(v_i) - c > 0$ (resp. < 0), where v_i is the ith vertex of the cube. The labels for the vertices, edges, and faces of the cube are respectively the followings.



The 256 entries of this table are grouped into 15 equivalence classes that result of the action of the direct product of the group of symmetries of the cube and group of permutations of the set $\{0, 1\}$. We refer to these classes as cases. The test table contains the information about the topological tests which are eventually performed on the faces and on the interior of a cube in which is present some kind of ambiguity. The table also stores the label of each face to be tested, and maps the results of those tests to the corresponding subcase. The solution of each subcase corresponds to one of the entries of lookup table presented by Nielson in [2].

Broadly speaking, the algorithm works as following:

For each cube of the grid,

1. Determine the case number and the configuration number.
2. Lookup which test should be performed for this configuration.
3. Determine the corresponding subcase based on the result of the test.
4. Lookup the tiling of the configuration for this subcase.

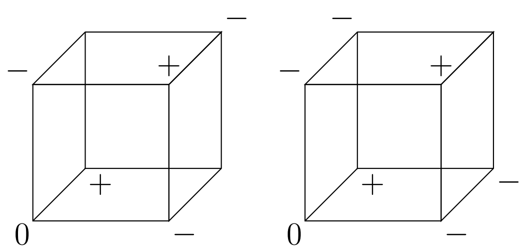
Our algorithm expands this by adding the possibility of assigning the value 0 to any vertex of the cube. Lewiner's algorithm, as well as all the versions of the original algorithm, assumes that the surface never passes through a vertex of the cube. A reason to do this is that when f is a truly real-valued function, the probability of the isosurface passing through a vertice of the grid is zero. However, when the function takes value in a countably set, the isosurface may touch many vertices of the grid with non-zero probability. So, taking into consideration the fact that in preactices we often find interger-valued isosurface leads to a much more robust algorithm with an extensive lookup table. The configuration table has $3^8 = 6561$ entries represented as 8-trit words in which a bit corresponding to the ith vertex v_i of the cube is 0, 1, 2 if $f(v_i) - c$ is zero, positive, and negative, respectively. In this case the group of equivalences produces 147 orbits instead of 15.

The test table does not require to be modified. Vertices with zeros does not lead to new internal ambiguities. This is formalized in the following lemma.

Lemma 7. *1. Let C be a signed square in which at least a vertex is zero. Then, C does not present ambiguity.*

2. Let C be a signed cube in which at least a vertex is zero. Then, C does not present internal ambiguity.

Proof. The proof follows from the fact that if two vertices can be connected through a tunnel, let us assume without restriction of generality that they are positive, then it is impossible to connect them by a sequence of positive or null vertices. Let us call this kind of sequence of *admissible path*, for short. Assume that A_0 and A_1 are two the endding points of an internal diagonal of the cube with positive sign. Let A_2 be a null vertice of the cube. Notice that in dimension two A_2 is adjancent to both vertices A_0 and A_1, and consequently it is always possible to connect these vertices by an admissible path. Now, in dimention three, A_2 is adjacent to only one of the vertices A_0 and A_1. To fix ideas, let us say that there is an edge of the cube connecting A_2 and A_0. If there is no admissible path connecteing A_0 and A_1, then we have one of the following configurations:



The reader can easily verify its own or with the aid of the list of solutions given at the end of the previous chapter that none of the possible solutions of these configurations presents a tunnel. □

The solution of each case and subcase correspond with some of the entries of the list exhibited in the last section of the previous chapter.

CHAPTER 3

Linear Interpolation in $\mathbb{R}^4$: A coarse classification

Here we proceed in a similar manner as we did in Chapter 1. Our main goal is to give a complete classification of the solutions of the linear interpolation equation:

$$\begin{aligned} P(x,y,z,t) = & b_0 + b_1x + b_2y + b_3z + b_4t + b_5xy + b_6xz + b_7xt + b_8yz + b_9yt \\ & + b_{10}zt + b_{11}xyz + b_{12}xyt + b_{13}xzt + b_{14}yzt + b_{15}xyzt = 0, \end{aligned} \tag{3.1}$$

where all coefficients are real numbers. The equivalences are defined as algebraic isomorphisms $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ given by $\Phi = \sigma \circ \varphi$, where $\varphi(x,y,z,t) = (ax+e, by+f, cz+g, dt+h)$ $(a,b,c,d \neq 0)$ is a diagonal isomorphism, and σ is a permutation of the set $\{x,y,z,t\}$. We said that two solutions S and S' of Equation (3.1) obtained from the polynomials P and P', respectively, are equivalent if and only if there exists an equivalence Φ such that $P' = CP \circ \Phi$, for some real constant C.

Let $P \circ \varphi(x,y,z,t) = B_0 + B_1x + B_2y + b_3z + B_4t + B_5xy + B_6xz + B_7xt + B_8yz + B_9yt + B_{10}zt + B_{11}xyz + B_{12}xyt + B_{13}xzt + B_{14}yzt + B_{15}xyzt$. We have the following relations:

$$
\begin{aligned}
B_0 &= b_0 + b_1e + b_2f + b_3g + b_4h + b_5ef + b_6eg + b_7eh + b_8fg + b_9fh + \\
&\quad b_{10}gh + b_{11}efg + b_{12}efh + b_{13}egh + b_{14}fgh + b_{15}efgh. && (3.2)\\
B_1 &= a(b_1 + b_5f + b_6g + b_7h + b_{11}fg + b_{12}fh + b_{13}gh + b_{15}fgh. && (3.3)\\
B_2 &= b(b_2 + b_5e + b_8g + b_9h + b_{11}eg + b_{12}eh + b_{14}gh + b_{15}egh). && (3.4)\\
B_3 &= c(b_3 + b_6e + b_8fb_{10}h + b_{11}ef + b_{13}eh + b_{14}fh + b_{15}efh). && (3.5)\\
B_4 &= d(b_4 + b_7e + b_9f + b_{10}g + b_{12}ef + b_{13}eg + b_{14}fg + b_{15}efg). && (3.6)\\
B_5 &= ab(b_5 + b_{11}g + b_{12}h + b_{15}gh). && (3.7)\\
B_6 &= ac(b_6 + b_{11}f + b_{13}h + b_{15}fh). && (3.8)\\
B_7 &= ad(b_7 + b_{12}f + b_{13}g + b_{15}fg). && (3.9)\\
B_8 &= bc(b_8 + b_{11}e + b_{14}h + b_{15}eh). && (3.10)\\
B_9 &= bd(b_9 + b_{12}e + b_{14}g + b_{15}eg). && (3.11)\\
B_{10} &= cd(b_{10} + b_{13}e + b_{14}f + b_{15}ef). && (3.12)\\
B_{11} &= abc(b_{11} + b_{15}h). && (3.13)\\
B_{12} &= abd(b_{12} + b_{15}g). && (3.14)\\
B_{13} &= acd(b_{13} + b_{15}f). && (3.15)\\
B_{14} &= bcd(b_{14} + b_{15}e). && (3.16)\\
B_{15} &= abcdb_{15}. && (3.17)
\end{aligned}
$$

Also as we noted earlier, the sequence of coefficients of $P \circ \sigma$ is a permutation of the sequence of coefficients of P. The following table contains the case of the generator permutations σ_{xy}, σ_{xz}, σ_{xt}, σ_{yz}, σ_{yt}, and σ_{zt}.

Table 3.1: Coefficient permutations.

P	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
$P \circ \sigma_{xy}$	b_0	b_2	b_1	b_3	b_4	b_5	b_8	b_9	b_6	b_7	b_{10}	b_{11}	b_{12}	b_{14}	b_{13}	b_{15}
$P \circ \sigma_{xz}$	b_0	b_3	b_2	b_1	b_4	b_8	b_6	b_{10}	b_5	b_9	b_7	b_{11}	b_{14}	b_{13}	b_{12}	b_{15}
$P \circ \sigma_{xt}$	b_0	b_4	b_2	b_3	b_1	b_9	b_{10}	b_7	b_8	b_5	b_6	b_{14}	b_{12}	b_{13}	b_{11}	b_{15}
$P \circ \sigma_{yz}$	b_0	b_1	b_3	b_2	b_4	b_6	b_5	b_7	b_8	b_{10}	b_9	b_{11}	b_{13}	b_{12}	b_{14}	b_{15}
$P \circ \sigma_{yt}$	b_0	b_1	b_4	b_3	b_2	b_7	b_6	b_5	b_{10}	b_9	b_8	b_{13}	b_{12}	b_{11}	b_{14}	b_{15}
$P \circ \sigma_{zt}$	b_0	b_1	b_2	b_4	b_3	b_5	b_7	b_6	b_9	b_8	b_{10}	b_{12}	b_{11}	b_{13}	b_{14}	b_{15}

3.1 The invariants

Next, we annunciate some important basic results which constitute the analogous of Lemmas 1-4 in dimension 4. To avoid tedious repetitions, we omite the proofs.

Lemma 8. *Let S be a solution of the equation $P(x, y, z, t) = 0$. The intersection of S with an arbitrary hiperplane normal to a principal axe is a solution of the tree-dimensional case.*

Lemma 9. *All the leaves of the foliation F_θ, except at most for three, are equally classified as a solution of a three-dimensional case.*

Lemma 10. *the foliations F_θ and $\varphi(F_\theta)$ are equally distributed.*

Corollary 5. *The distribution of the foliation F_θ of S is an invariant.*

Corollary 6. *The combination of distributions of the four foliations of S is an invariant.*

3.2 A bit of algebra

The goal of this section is to analyze the solutions of a linear system whose coefficients are polynomials of degree 1. In other words, we wish to know whether a system

$$\begin{cases} (A + Bx) + Y(C + Dx) = 0 \\ (E + Fx) + Y(G + Hx) = 0 \end{cases} \tag{3.18}$$

in the variable Y is compatible or not by attending not only to Y, but the values of the parameter x. The results that we shall present in this section play a main role in the analysis that we will perform in the next section.

By virtue of the Kronecker-Capelli's Theorem, we can assert that this system is compatible if and only if the matrixes

$$\Phi(x) = \begin{pmatrix} C + Dx \\ G + Hx \end{pmatrix} \text{ and } \widetilde{\Phi}(x) = \begin{pmatrix} A + Bx & C + Dx \\ E + Fx & G + Hx \end{pmatrix} = \begin{pmatrix} \Gamma(x) & \Phi(x) \end{pmatrix}$$

have the same rank.

For each value of x, the $\operatorname{rank}\Phi(x)$ is either 0 or 1. Since all non-identically null coefficient of $\Phi(x)$ vanishes at most once, we have the following alternatives for $\operatorname{rank}\Phi(x)$:

1. $\text{rank}\Phi(x) = 0$, for all $x \in \mathbb{R}$. This happens if and only if $C = D = G = H = 0$
2. $\text{rank}\Phi(x) = 1$, for all $x \in \mathbb{R} - \{x_0\}$, and $\text{rank}\Phi(x_0)$ is 0. This occurs if and only if x_0 satisfies both equations $C + xD = 0$ and $G + xH = 0$. This amounts to say that at least one of the coefficients D and H is non-null, and there is $\alpha \in \mathbb{R}$ such that $(C, D) = \alpha(G, H)$.
3. $\text{rank}\Phi(x) = 1$, for all $x \in \mathbb{R}$. This alternative arises if and only if there is no real value satisfying simultaneously the equations $C + xD = 0$ and $G + xH = 0$. This equivals to say that at least one of the coefficients C and G is non-null, and for all $\alpha \in \mathbb{R}$, $(C, D) \neq \alpha(G, H)$.

Suppose that $\text{rank}\Phi(x) = 0$ for all $x \in \mathbb{R}$. The $\text{rank}\widetilde{\Phi}(x)$ can be:

(a) 0, for all value of $x \in \mathbb{R}$. In this case $A = B = F = I = 0$, and System (3.18) is compatible for all real value of x.

(b) 1, for all $x \in \mathbb{R} - \{x_1\}$, and $\text{rank}\widetilde{\Phi}(x_1) = 0$. In this case there exists $\beta \in \mathbb{R}$ such that $A = \beta B$ and $F = \beta I$, and System (3.18) is only compatible for $x = x_1$.

(c) 1, for all $x \in \mathbb{R}$. This situation emerges when none of the conditions above holds, and in this case System (3.18) is incompatible for any real value of x.

Now assume that $\text{rank}\Phi(x) = 1$ for all $x \in \mathbb{R} - \{x_0\}$, and $\text{rank}\Phi(x_0) = 0$. This gives the following possibilities for $\text{rank}\widetilde{\Phi}(x)$:

(a) $\text{rank}\widetilde{\Phi}(x) = 1$ for all $x \in \mathbb{R} - \{x_0\}$, and $\text{rank}\widetilde{\Phi}(x_0) = 0$. This happens if and only if there exists a real number β such that $\Gamma(x) = \beta\Phi(x)$. In this case System (3.18) is compatible for all real value of x.

(b) $\text{rank}\widetilde{\Phi}(x) = 1$ for all $x \in \mathbb{R}$. This situation appears if and only $\Gamma(x)$ is constant, and there exists a non-constant linear function $g(x)$ such that $\Phi(x) = g(x)\Gamma(x)$. In this case System (3.18) is compatible for all $x \in \mathbb{R} - \{x_0\}$, and incompatible for $x = x_0$.

(c) $\text{rank}\widetilde{\Phi}(x) = 2$ for all $x \in \mathbb{R} - \{x_0\}$, and $\text{rank}\widetilde{\Phi}(x_0) = 1$. This situation emerges if and only if the coefficients of $\Gamma(x)$ do not vanish at the same time. In this case the system (3.18) is incompatible for any value of x.

(d) rank$\widetilde{\Phi}(x) = 2$ for all $x \in \mathbb{R} - \{x_0, x_1\}$, $x_0 \neq x_1$, and rank$\widetilde{\Phi}(x_0) =$ rank$\widetilde{\Phi}(x_1) = 1$. This occurs if and only if the coefficients of $\Gamma(x)$ vanish at x_1 and the coefficients of $\Phi(x)$ do it at x_0. In this case System (3.18) is uniquely compatible for $x = x_1$.

Finally, consider that rank$\Phi(x) = 1$ for all $x \in \mathbb{R}$. The possibilities for rank$\widetilde{\Phi}(x)$ are the following:

(a) rank$\widetilde{\Phi}(x) = 1$ for all $x \in \mathbb{R}$. This situation arises when there exists a real constant β such that $\Gamma(x) = \beta\Phi(x)$. In this case System (3.18) is compatible for all real value of x.

(b) rank$\widetilde{\Phi}(x) = 2$ for all $x \in \mathbb{R}$. This occurs if and only if for any linear function $g(x)$, $\Gamma(x) \neq \Phi(x)$. In this case System (3.18) is incompatible for all real value of x.

(c) rank$\widetilde{\Phi}(x) = 2$, for all $x \in \mathbb{R} - \{x_0\}$, and rank$\widetilde{\Phi}(x_0) = 1$. This occurs if and only if the coefficients of $\Gamma(x)$ vanish at x_0. In this case System (3.18) is only compatible for $x = 0$.

Summarizing, we have that System (3.18) is:

1. compatible for any real value of x: rank$\Phi(x) = 0$ for all $x \in \mathbb{R}$, situation (a); rank$\Phi(x) = 1$ for all $x \in \mathbb{R} - \{x_0\}$, situation (a); rank$\Phi(x) = 1$ for all $x \in \mathbb{R}$, situation (a).

2. compatible for any real value of x, except for a unique value: rank$\Phi(x) = 1$ for all $x \in \mathbb{R} - \{x_0\}$, situation (b).

3. compatible for a unique value of x: rank$\Phi(x) = 0$ for all $x \in \mathbb{R}$, situation (b); rank$\Phi(x) = 1$ for all $x \in \mathbb{R} - \{x_0\}$, situation (d); rank$\Phi(x) = 1$ for all $x \in \mathbb{R}$, situation (c).

4. incompatible for any real value of x: rank$\Phi(x) = 0$ for all $x \in \mathbb{R}$, situation (c); rank$\Phi(x) = 1$ for all $x \in \mathbb{R} - \{x_0\}$, situation (c); rank$\Phi(x) = 1$ for all $x \in \mathbb{R}$, situation (b).

3.3 Foliation in 4D: a coarse classification

Having disposed of these preliminary steps, we are now able to introduce the detailed list of all possibilities for F_x. The characterization of each possibility is based on the factorization

$$\begin{aligned} P(x,y,z,t) =& (b_0 + b_1 x) + (b_2 + b_5 x)y + (b_3 + b_6 x)z + (b_4 + b_7 x)t \\ &+ (b_8 + b_{11} x)yz + (b_9 + b_{12})yt + (b_{10} + b_{13})zt + (b_{14} + b_{15} x)yzt, \end{aligned}$$

which, in order to use the classification of the three-dimensional case, we rewrite in the three following forms:

$$[(b_0 + b_1x) + y(b_2 + b_5x)] + [(b_3 + b_6x) + y(b_8 + b_{11}x)]z + [(b_4 + b_7x) + y(b_9 + b_{12}x)]t + [(b_{10} + b_{13}x) + y(b_{14} + b_{15}x)]zt.$$
$$[(b_0 + b_1x) + z(b_3 + b_6x)] + [(b_2 + b_5x) + z(b_8 + b_{11}x)]y + [(b_4 + b_7x) + z(b_{10} + b_{13}x)]t + [(b_9 + b_{12}x) + z(b_{14} + b_{15}x)]yt.$$
$$[(b_0 + b_1x) + t(b_4 + b_7x)] + [(b_2 + b_5x) + t(b_9 + b_{12}x)]y + [(b_3 + b_6x) + t(b_{10} + b_{13}x)]z + [(b_8 + b_{11}x) + t(b_{14} + b_{15}x)]yz.$$

Each possibility is described by attending to the classification of its leaves as solutions of the three-dimensional case and using the result presented in the previous section. Because the analysis of all cases is very extensive, we shall exibit the first cases in order to illustrates the procedure. The remaining ones can be found in Appendix.

Predominance of $C1$

P1 All the leaves being $C1$. This only occurs when $b_0 \neq 0$, and all remaining coefficients are null.

P2 All the leaves being $C1$, except for one of them which is $C11$. This case appears when $b_1 \neq 0$, and for all $i \geq 2$, $b_i = 0$. (There is no restriction on b_0.)

Predominance of $C2$

P3 All the leaves being $C2$. This case emerges when $b_8 = b_9 = b_{10} = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, and at least two of the conditions (a) $b_2 \neq 0$ and $b_5 = 0$, (b) $b_3 \neq 0$ and $b_6 = 0$, and (c) $b_4 \neq 0$ and $b_{11} = 0$ are satisfied.

P4 All the leaves being $C2$, except for one of them which is $C1$. This occurs when $b_8 = b_9 = b_{10} = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, and there exists a value x_0 such that $b_2 + b_5x_0 = b_3 + b_6x_0 = b_4 + b_7x_0 = 0$, but $b_0 + b_1x_0 \neq 0$.

P5 All the leaves being $C2$, except for one of them which is $C11$. This occurs when $b_8 = b_9 = b_{10} = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, and there exists a value x_0 such that $b_0 + b_1x_0 = b_2 + b_5x_0 = b_3 + b_6x_0 = b_4 + b_7x_0 = 0$.

Predominance of $C3$

P6 All the leaves being $C3$. This case appears when $b_7 = b_9 = b_{10} = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, and b_4 and b_8 are non-null.

P7 All the leaves being $C3$, except for one of them which is $C2$. $b_7 = b_9 = b_{10} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, $b_4 \neq 0$, and there exists an only value x_0 such that $b_8 + b_{11}x_0 = 0$, and either $b_2 + b_5x_0 \neq 0$ or $b_3 + b_6x_0 \neq 0$.

P8 All the leaves being $C3$, except for one of them which is $C4$. This occurs when $b_7 = b_9 = b_{10} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, $b_4 \neq 0$, and there exists a unique value x_0 such that $b_2 + b_5x_0 = b_3 + b_6x_0 = b_8 + b_{11}x_0 = 0$.

P9 All the leaves being $C3$, except for one of them which is $C9$. This possibility emerges when $b_4 = b_7 = b_9 = b_{10} = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, $b_8 \neq 0$, and there exists an only value x_0 such that $b_4 + b_7x_0 = 0$, and the equations $b_6x_0 + Xb_8 = 0$ and $(b_2 + b_5x_0) + Xb_8 = 0$ have respective solutions which do not satisfy the equations $(b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0$ and $(b_0 + b_1x_0) + Xb_6x_0 = 0$, respectively.

P10 All the leaves being $C3$, except for one of them which is $C10$. This possibility emerges when $b_9 = b_{10} = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, $b_8 \neq 0$, and there exists an only value x_0 such that $b_4 + b_7x_0 = 0$, and the equations $b_6x_0 + Xb_8 = 0$ and $(b_2 + b_5x_0) + Xb_8 = 0$ have respective solutions which satisfy the equations $(b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0$ and $(b_0 + b_1x_0) + Xb_6x_0 = 0$, respectively.

P11 All the leaves being $C3$, except for two of them of which one is $C4$ and the other is $C9$. This occurs when $b_9 = b_{10} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, and there exist different values x_0 and x_1 such that: (a) $b_4 + b_7x_0 \neq 0$, and $b_2 + b_5x_0 = b_3 + b_6x_0 = b_8 + b_{11}x_0 = 0$, and (b) $b_8 + b_{11}x_1 \neq 0$, $b_4 + b_7x_1 = 0$, and the equations $(b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0$ and $(b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0$ have respective solutions which do not satisfy the equations $(b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0$ and $(b_0 + b_1x_1) + X(b_3 + b_6x_1) = 0$, respectively.

P12 All the leaves being $C3$, except for two of them of which one is $C4$ and the other is $C10$. This possibility emerges when $b_9 = b_{10} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, and there exist different values x_0 and x_1 such that: (a) $b_4 + b_7x_0 \neq 0$, and $b_2 + b_5x_0 = b_3 + b_6x_0 = b_8 + b_{11}x_0 = 0$, and (b) $b_8 + b_{11}x_1 \neq 0$, $b_4 + b_7x_1 = 0$, and the equations $(b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0$ and $(b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0$ have respective solutions which satisfy the equations $(b_0+b_1x_1)+X(b_2+b_5x_1)=0$ and $(b_0+b_1x_1)+X(b_3+b_6x_1)=0$, respectively.

P13 All the leaves being $C3$, except for two of them of which one is $C2$ and the other is $C9$. This occurs when $b_9=b_{10}=b_{12}=b_{13}=b_{14}=b_{15}=0$, and there exist distinct values x_0 and x_1 such that: (a) $b_4+b_7x_0\neq 0$, $b_8+b_{11}x_0=0$, and either $b_2+b_5x_0\neq 0$ or $b_3+b_6x_0\neq 0$, and (b) $b_8+b_{11}x_1\neq 0$, $b_4+b_7x_1=0$, and the equations $(b_3+b_6x_1)+X(b_8+b_{11}x_1)=0$ and $(b_2+b_5x_1)+X(b_8+b_{11}x_1)=0$ have respective solutions which do not satisfy the equations $(b_0+b_1x_1)+X(b_2+b_5x_1)=0$ and $(b_0+b_1x_1)+X(b_3+b_6x_1)=0$, respectively.

P14 All the leaves being $C3$, except for two of them, one of which is $C2$ and the other is $C9$. This case appears when $b_9=b_{10}=b_{12}=b_{13}=b_{14}=b_{15}=0$, and there exist distinct values x_0 and x_1 such that: (a) $b_4+b_7x_0\neq 0$, $b_8+b_{11}x_0=0$, and either $b_2+b_5x_0\neq 0$ or $b_3+b_6x_0\neq 0$, and (b) $b_8+b_{11}x_1\neq 0$, $b_4+b_7x_1=0$, and the equations $(b_3+b_6x_1)+X(b_8+b_{11}x_1)=0$ and $(b_2+b_5x_1)+X(b_8+b_{11}x_1)=0$ have respective solutions which satisfy the equations $(b_0+b_1x_1)+X(b_2+b_5x_1)=0$ and $(b_0+b_1x_1)+X(b_3+b_6x_1)=0$, respectively.

3.4 A primary marching hypercubes algorithm

This section introduces a basic algorithm to solve the problem in dimension four.

3.4.1 General description

The algorithm we present here is restricted to configurations not presenting null vertices nor internal ambiguities. Roughly speaking, the algorithm proceeds as following:

For each hypercube of the grid,

1. Determine the case number and configuration.
2. Lookup which test should be performed for this configuration.
3. Determine the corresponding subcase based on the result of the test.

4. Lookup the tiling of the configuration for this subcase.

The algorithm is mainly based on three tables: the configuration table, the test table, and the tiling table. The configuration table stores the $2^{16} = 65536$ possible configurations for a hypercube in a matrix whose entries are words of 16 bits, one for each vertex of the hypercube. Each bit can be 0 or 1 accordingly to the sign of f at the corresponding vertex. In this case the group of symmetries and the group of permutations of the set $\{0, 1\}$ lead to 222 orbits. We refer to these orbits as cases. In other words, a clase is a collection of equivalent hypercube configurations. The test table stores for each case the test to be performed to resolve topological ambiguity in each configuration. This table also maps the result of each test into the corresponding subcase. The tiling table stores the solution for each subcase. For this primary algorithm, the hypercubes are tiled by triangles that form tetrahedra. For visualization purposes, the vertices of the triangle are either middle points of the edges of the hypercube or a middle point of a three-dimensional face of the hypercube. The input of the algorithm is a four-dimensional signed grid over which the scalar field f was sampled; and the output is a triangulated surface homeomorphic to $F^{-1}(c)$, where F matches f at the vertex of G and is 4-linear within each hypercube.

3.4.2 Constructing the first and second tables

The configuration table is constructed by using GAP (Groups, Algorithms, and Programming), a free software containing numerical packages for computacional group theory [27].

The first step is to create the shapeGroup, colorGroup, and coloringGroup for dimension four and two colors. Here the word colors is used to mean the signs of the vertices of the hypercube.

```
n:=4;; // 4 dimensions
k:=2;; // 2 classes (+,-)
shapeGroup := WreathProductProductAction(SymmetricGroup(2),
   SymmetricGroup(n));;
colorGroup := Group (PermList (Reversed ([1..k])));;
coloringGroup := DirectProduct (shapeGroup, colorGroup);;
```

Once the result is obtained, we build projection operators to extract these groups back from their direct product.

```
shapeProjection := Projection (coloringGroup, 1);;
colorProjection := Projection (coloringGroup, 2);;
```

The next step is to obtain the list of colors and colorings, taking the result from GAP.

```
numVerts := 2^n;;
coloredVerts := ListWithIdenticalEntries (numVerts, [1..k]);
[ [1..2], [1..2], [1..2], [1..2], [1..2], [1..2], [1..2], [1..2],
[1..2], [1..2], [1..2], [1..2], [1..2], [1..2], [1..2], [1..2] ]
colorings:= Cartesian (coloredVerts);
```

Now, we use the function below to produce the action of the group element on a coloring.

```
action := \textbf{function} (coloring, groupElement)
  \textbf{local} shapePerm, shuffled, result;
  shapePerm := Image (shapeProjection, groupElement);
  colorPerm  := Image (colorProjection, groupElement);
  shuffled     := Permuted (coloring, shapePerm);
  result         := On Tuples (shuffled, colorPerm);
  \textbf{return} result;
\textbf{end};;
```

Finally, we use gap to obtain the orbits of the previous action.

```
orbits := OrbitsDomain (coloringGroup, colorings, action);
```

To construct the second table we store the label of the ambiguous bi-dimensional faces of each configuration and considered all the possible combinations of signs for such faces. This leads to a huge table which is shortened when the tiling table is created in the following way: if for some combination of signs of the ambigue faces of a given configuration we are not able to exibit a solution, then this combination of signs is removed from the table.

3.4.3 Constructing the third table

To construct the tiling table we splite the hypercube into its 8 three-dimensional faces and used the Lewiner's algorithm (see Chapter 3, Section 1) to obtain a tiling of each one of these faces. We denote the ith three-dimensional face of the hypercube by F_i. The respective correspondences between the vertices, edges, and faces of each F_i and the vertices, edges, and faces of the three-dimensional cube are detailed in Tables 3.2, 3.3, and 3.4, respectively. Once the solution of each of the eigth three-dimentional faces is produced, we concatenate them removing repeated triangles. It is worthy to mention that in some cases the tetrahedralization is not complete becuase connections between vertices belonging to different three-dimensional faces have not yet been exploited.

Table 3.2: Correspondence between vertices.

	0	1	2	3	4	5	6	7
F1	0	1	9	8	2	3	11	10
F2	4	5	13	12	6	7	15	14
F3	4	0	8	12	6	2	10	14
F4	1	5	13	9	3	7	15	11
F5	4	5	13	12	0	1	9	8
F6	2	3	11	10	6	7	15	14
F7	8	9	13	12	10	11	15	14
F8	0	1	5	4	2	3	7	6

Table 3.3: Correspondence between edges.

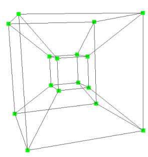
	0	1	2	3	4	5	6	7	8	9	10	11	12
F1	0	17	8	16	2	18	10	19	3	1	9	11	32
F2	4	21	12	20	6	22	14	23	7	5	13	15	33
F3	24	16	28	20	25	19	29	23	7	3	11	15	34
F4	26	21	30	17	27	22	31	18	1	5	13	9	35
F5	4	21	12	20	0	17	8	16	24	26	30	28	36
F6	2	18	10	19	6	22	14	23	25	27	31	29	37
F7	8	30	9	28	10	31	14	29	11	9	13	15	38
F8	0	26	4	24	2	27	6	25	3	1	5	7	39

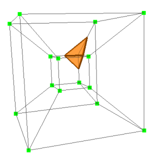
Table 3.4: Correspondence between faces.

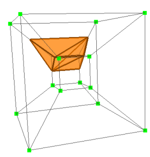
	0	1	2	3	4	5
F1	6	2	5	1	3	4
F2	12	7	11	8	9	10
F3	16	1	15	8	13	14
F4	20	7	19	2	17	18
F5	22	17	21	13	9	3
F6	24	18	23	14	4	10
F7	5	19	11	15	21	23
F8	6	20	12	16	22	24

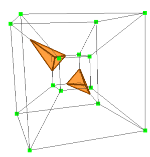
3.4.4 The hypercube's cases

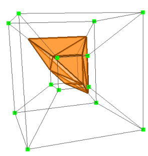
In this section we show the tiling for the configuration 0 of each case.

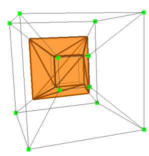

Case 0

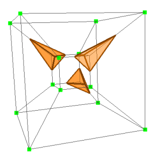

Case 1

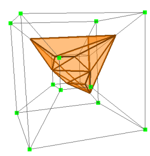

Case 2


Case 3


Case 4


Case 5


Case 6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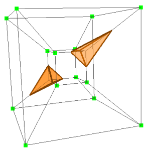

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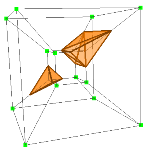

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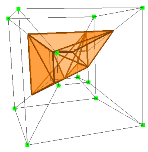

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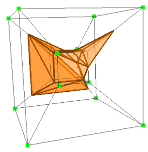

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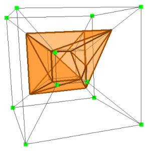

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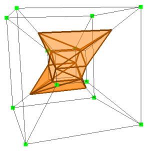

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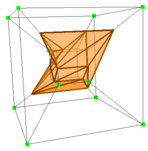

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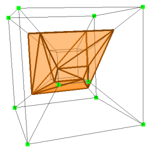

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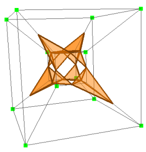

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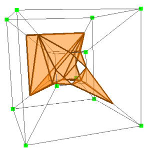

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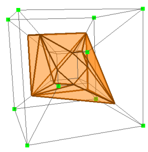

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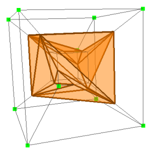

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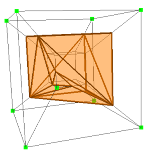

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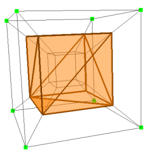

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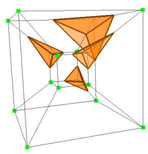

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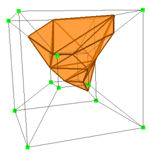

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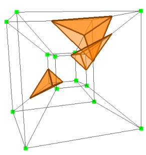

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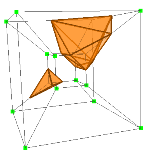

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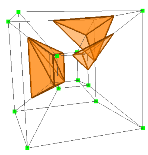

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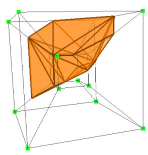

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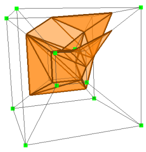
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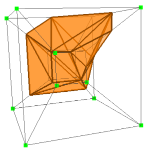
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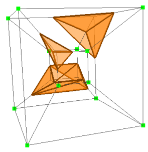
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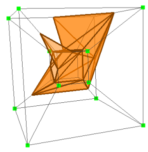
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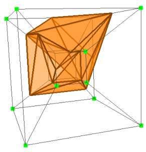
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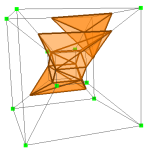
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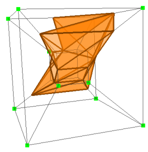
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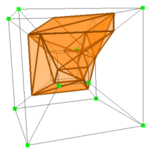
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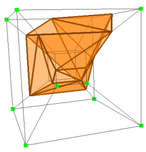
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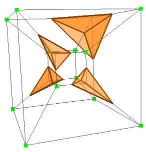
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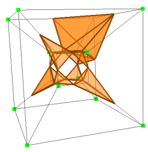
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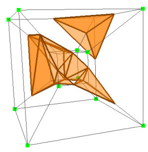
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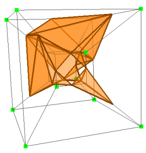
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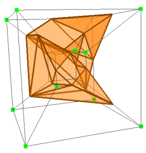
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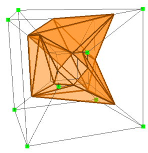
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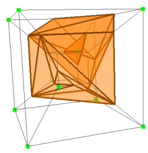
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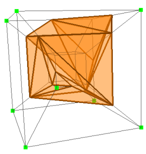
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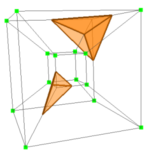
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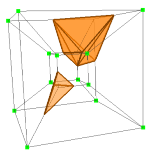
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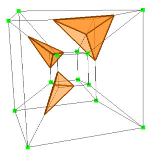
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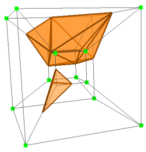
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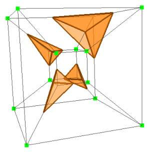
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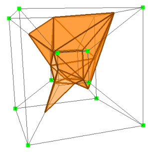
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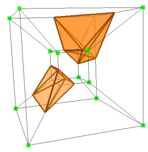
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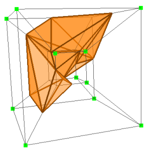
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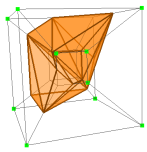
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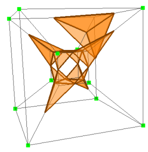
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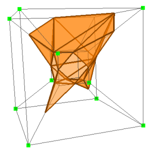
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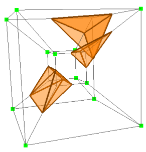
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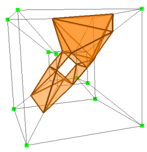
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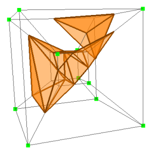
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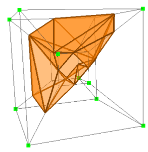
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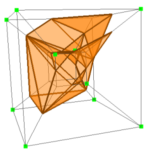
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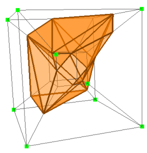
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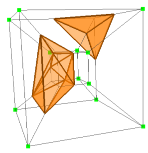
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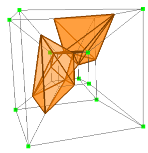
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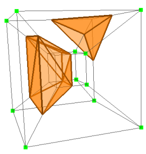
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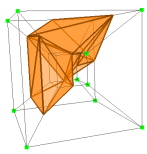
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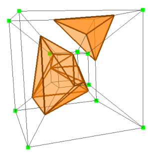
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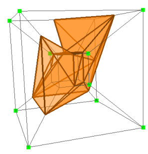
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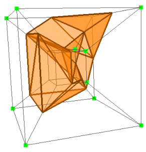
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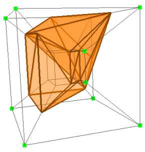
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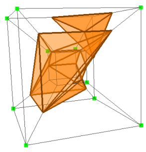
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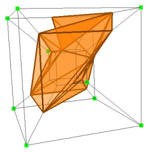
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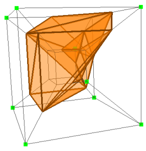
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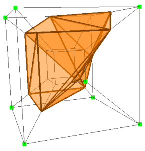
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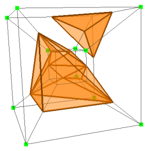
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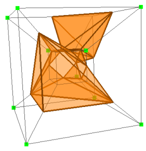
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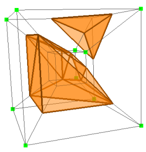
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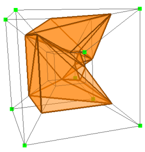
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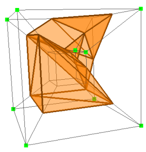
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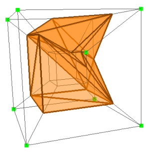
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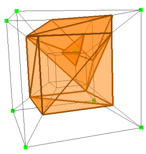
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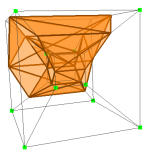
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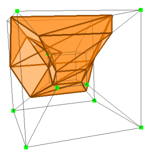
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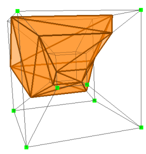
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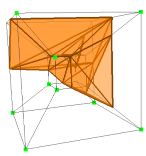
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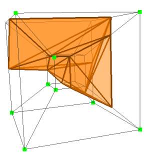
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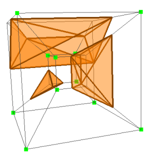
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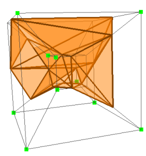
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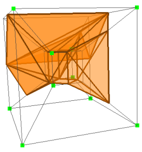
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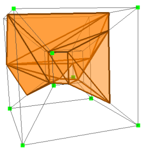
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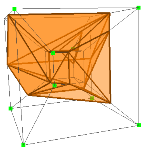
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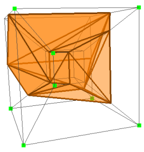
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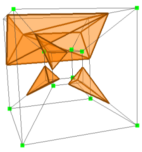
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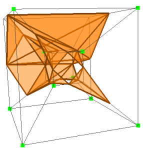
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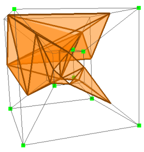
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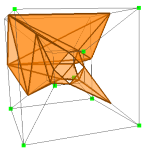
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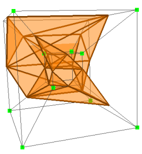
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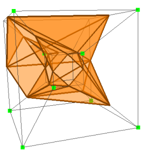
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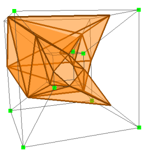
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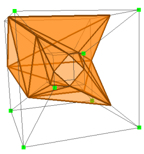
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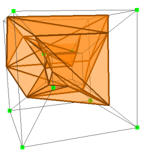
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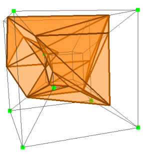
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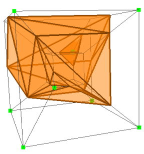
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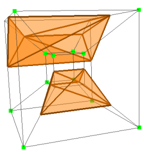
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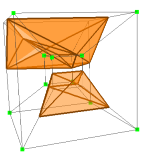
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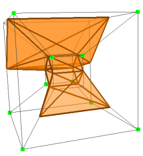
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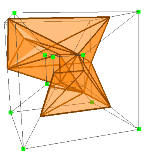
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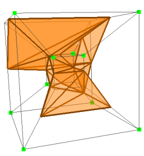
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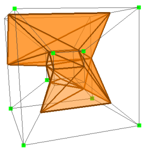
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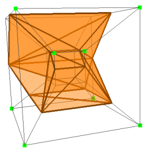
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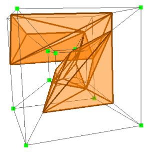
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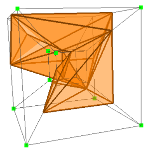
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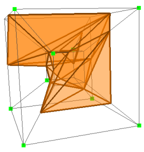
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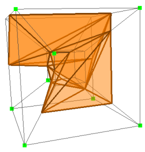
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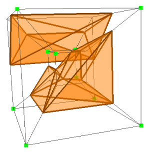
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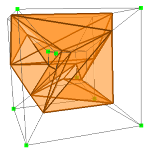
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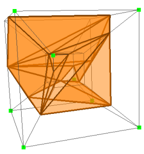
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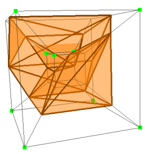
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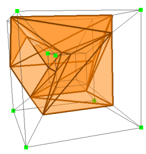
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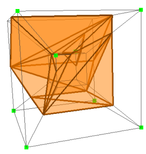
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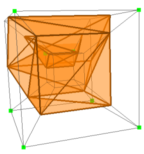
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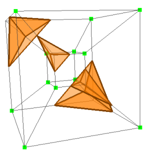
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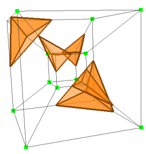
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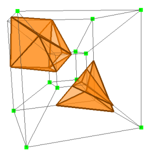
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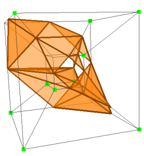
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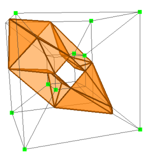
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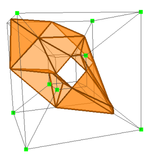
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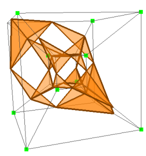
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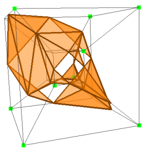
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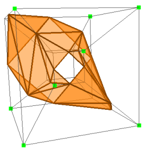
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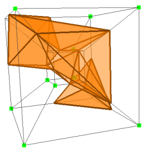
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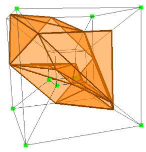
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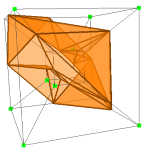
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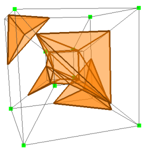
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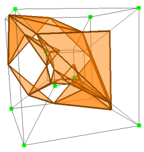
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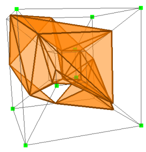
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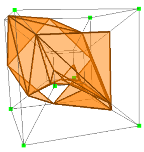
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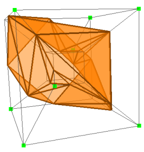
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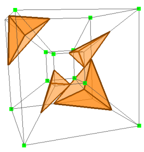
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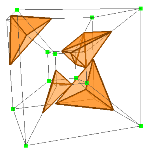
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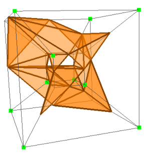
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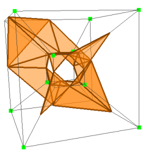
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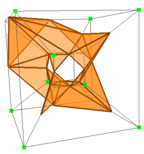
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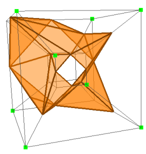
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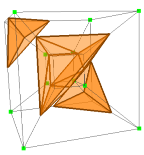
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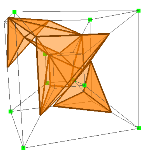
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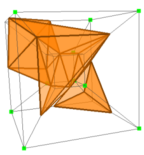
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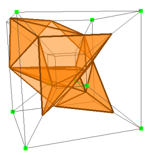
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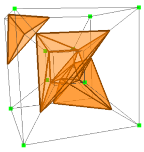
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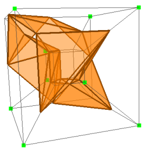
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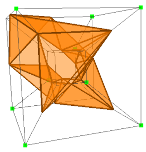
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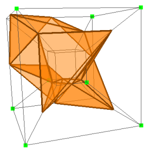
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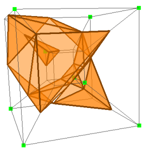
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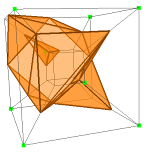
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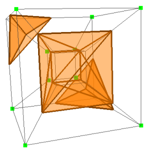
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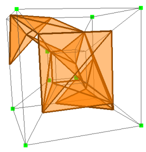
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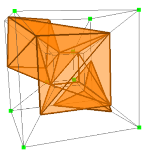
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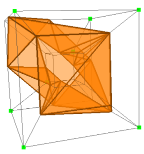
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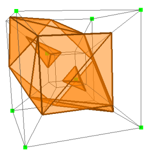
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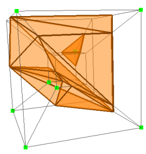
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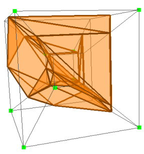
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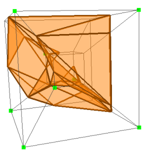
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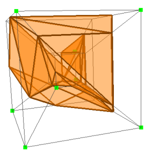
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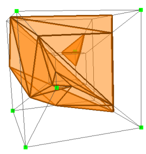
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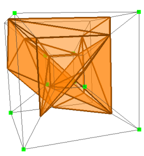
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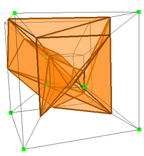
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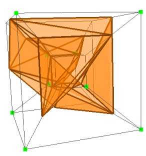
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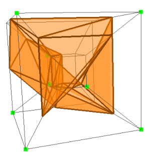
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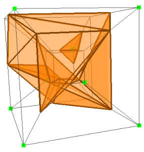
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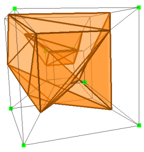
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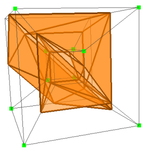
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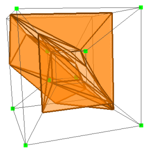
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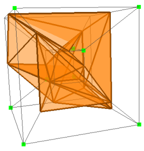
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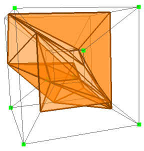
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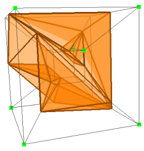
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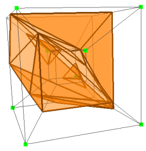
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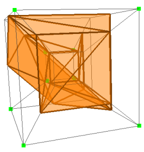
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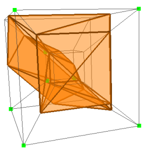
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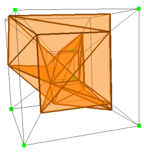
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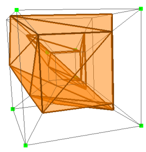
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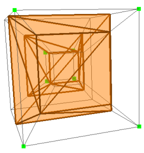
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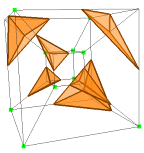
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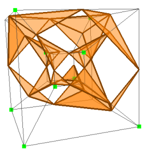
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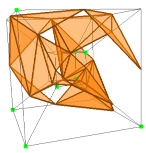
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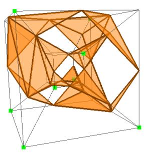
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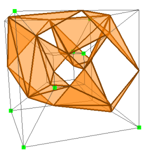
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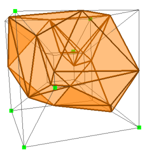
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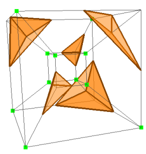
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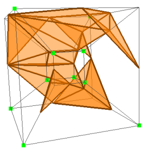
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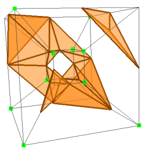
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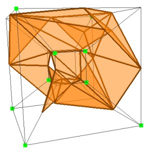
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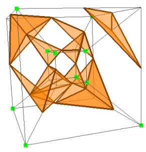
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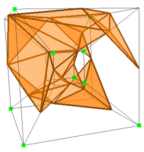
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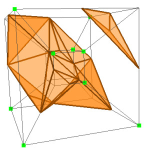
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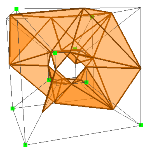
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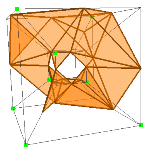
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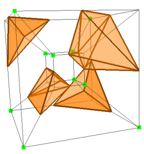
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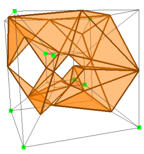
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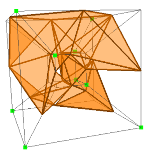
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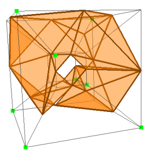
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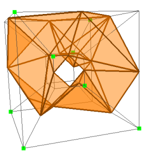
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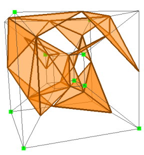
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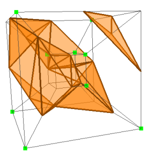
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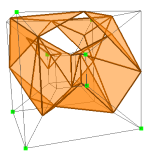
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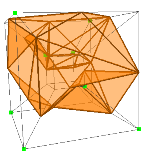
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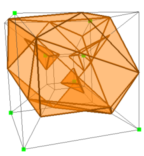
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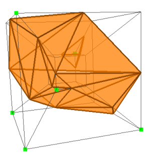
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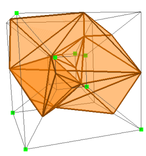
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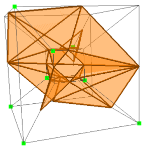
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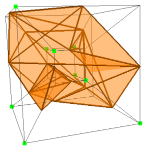
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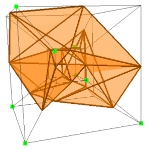
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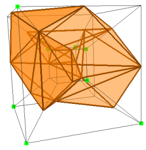
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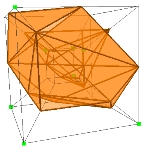
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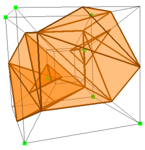
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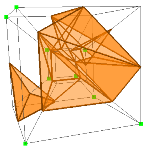
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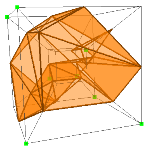
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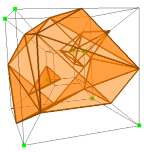
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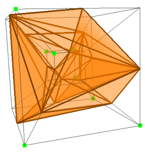
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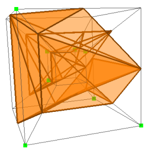
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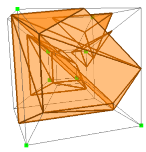
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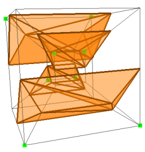
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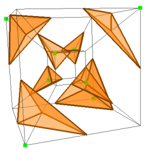
Case 221

Concluding Remarks and Future Works

This thesis presented two new algorithms for constructing isosurface in dimensions three and four which have as guiding philosophy the marching cubes methodology with our proposal of analysis through foliations. To achieve efficient algorithms, we needed to study the topology of the isosurfaces associated to n-linear functions, for $n = 3, 4$. In dimension three, we were able to present a complete classification of such isosurfaces, which was extremely useful to determine all possible cases for the improved algorithm. This contribution solves the problem in dimension three, at least from the point of view of robustness. Improvements to the algorithm in order to accelerate the execution time are always welcome

In four dimensions the situation is somewhat different. We took the first steps in this open field using foliations. We show a partial classification of the isohypersurface and introduce a primary algorithm which we plan to expand and improve in a short time. The principal drawback of the algorithm in the current form lies in the way that the tiling table was constructed. For each configuration of signs of the hypercube, we obtain the solution of each one of its three-dimensional faces by using the Lewiner's version of the marching cubes algorithm. The solution for the hypercube is given by concatenating these partial solutions, eliminating at the same time recurrent triangles. In some occasions this way produces an incomplete tetrahedralization of the solution.

To solve this problem, our major challenge is to obtain a complete classification of the solution of the equation $f(x, y, z, t) = 0$, where f is a fourth-linear function. We have obtained the basic classification by attending to the foliation that the solution produces when is intersected with planes perpendicular to a given axis. By studying how the possible foliations can be combined together in a same solution, we will have the complete classification.

With the classification at hand, we would have a representative solution for each marching hypercube case and thereby we will able to produce a correct tetrahedralization for it.

Appendix

Predominance of $C4$

P15 All the leaves being $C4$. This possibility arises when for all $i \geq 2$, $i \neq 4$, $b_i = 0$. (There is no restriction on b_0 and b_1).

P16 All the leaves being $C4$, except for one which is $C1$. This case emerges when b_0 and b_7 are non-null, there is no restriction on b_4 and b_1, all remaining coefficients are null, and the system $b_0 + b_1 x = b_4 + b_7 x = 0$ is incompatible.

P17 All the leaves being $C4$, except for one which is $C11$. This case emerges when b_1 and b_7 are non-null, there is no restriction on b_0 and b_4, all remaining coefficients are null, and the system $b_0 + b_1 x = b_4 + b_7 x = 0$ is compatible.

Predominance of $C5$

P18 All the leaves being $C5$. This case arises when $b_{10} = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, b_8 and b_9 are non-null, and the system

$$\begin{cases} (b_3 + b_6 x) + X b_8 = 0 \\ (b_4 + b_7 x) + X b_9 = 0 \end{cases}$$

is incompatible for any value of x.

P19 All the leaves being $C5$, except for one of them which is $C7$. This case arises when $b_{10} = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, b_8 and b_9 are non-null, and there exists a value x_0 such that the system $\begin{cases} (b_3 + b_6x_0) + Xb_8 = 0 \\ (b_4 + b_7x_0) + Xb_9 = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_2 + b_5x_0)$.

P20 All the leaves being $C5$, except for one of them which is $C8$. This case arises when $b_{10} = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, b_8 and b_9 are non-null, and there exists a value x_0 such that the system $\begin{cases} (b_3 + b_6x_0) + Xb_8 = 0 \\ (b_4 + b_7x_0) + Xb_9 = 0 \\ (b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0 \end{cases}$ is compatible.

P21 All the leaves being $C5$, except for one of them which is $C3$. This occurs when $b_{10} = b_{11} = b_{13} = b_{14} = b_{15} = 0$, b_8 and b_{12} are non null, the system $\begin{cases} (b_3 + b_6x) + Xb_8 = 0 \\ (b_4 + b_7x) + X(b_9 + b_{12}x_0) = 0 \end{cases}$ is incompatible for any value of x, and the system $\begin{cases} (b_3 + b_6x_0) + Xb_8 = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}) = 0 \end{cases}$ is incompatible, where x_0 is the solution of the equation $b_9 + b_{12}x = 0$.

P22 All the leaves being $C5$, except for one of them which is $C9$. This occurs when $b_{10} = b_{11} = b_{13} = b_{14} = b_{15} = 0$, b_8 and b_{12} are non null, the system $\begin{cases} (b_3 + b_6x) + Xb_8 = 0 \\ (b_4 + b_7x) + X(b_9 + b_{12}) = 0 \end{cases}$ is incompatible for each x different from the solution x_0 of the equation $b_9 + b_{12}x = 0$, and the system $\begin{cases} (b_3 + b_6x_0) + Xb_8 = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}) = 0 \end{cases}$ is compatible. It is also required that the respective solutions of the previous systems do not satisfy the equations $(b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0$ and $(b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0$, respectively.

P23 All the leaves being $C5$, except for one of them which is $C9$. This occurs when $b_{10} = b_{11} = b_{13} = b_{14} = b_{15} = 0$, b_8 and b_{12} are non null, the system $\begin{cases} (b_3 + b_6x) + Xb_8 = 0 \\ (b_4 + b_7x) + X(b_9 + b_{12}) = 0 \end{cases}$ is incompatible for each x different from the solution x_0 of the equation $b_9 + b_{12}x = 0$, and the system $\begin{cases} (b_3 + b_6x_0) + Xb_8 = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}) = 0 \\ (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \end{cases}$ is compatible. It is also required that the solution of the first system satisfy the equations $(b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0$.

P24 All the leaves being $C5$, except for two of them which are $C3$. This occurs when $b_{10} = b_{13} = b_{14} = b_{15} = 0$, b_{11} and b_{12} are non null, the system $\begin{cases} (b_3 + b_6x) + X(b_8 + b_{11}x) = 0 \\ (b_4 + b_7x) + X(b_9 + b_{12}) = 0 \end{cases}$ is incompatible for any value of x, and there exist distinct values x_0 and x_1 such that: (a) $b_8 + b_{11}x_0 = 0$, but $b_3 + b_6x_0 \neq 0$; and (b) $b_9 + b_{12}x_1 = 0$, but $b_4 + b_7x_1 \neq 0$.

P25 All the leaves being $C5$, except for one of them which is $C2$. This occurs when $b_{10} = b_{13} = b_{14} = b_{15} = 0$, b_{11} and b_{12} are non null, the system $\begin{cases} (b_3 + b_6x) + X(b_8 + b_{11}x) = 0 \\ (b_4 + b_7x) + X(b_9 + b_{12}) = 0 \end{cases}$ is incompatible for any value of x, and there exists a value x_0 such that $b_8 + b_{11}x_0 = b_9 + b_{12}x_0 = 0$, but $b_3 + b_6x_0$ and $b_4 + b_7x_0$ are non-null.

P26 All the leaves being $C5$, except for two of them, one of which is $C3$ and the other is $C9$. This occurs when $b_{10} = b_{13} = b_{14} = b_{15} = 0$, b_7, b_{11} and b_{12} are non null, and there are different values x_0 and x_1 such that: (a) $b_4 + b_7x_0 = b_9 + b_{12}x_0 = 0$, and the systems

$$\begin{cases} (b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0 \\ (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \end{cases}$$

and

$$\begin{cases} (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \\ (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \end{cases}$$

are incompatible; and (b) $b_8 + b_{11}x_1 = 0$, but $b_3 + b_6x_1 \neq 0$.

P27 All the leaves being $C5$, except for two of them, one of which is $C3$ and the other is $C10$. This case arises when $b_{10} = b_{13} = b_{14} = b_{15} = 0$, b_7, b_{11} and b_{12} are non null, and there are different values x_0 and x_1 such that: (a) $b_4 + b_7x_0 = b_9 + b_{12}x_0 = 0$, and the systems

$$\begin{cases} (b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0 \\ (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \end{cases}$$

and

$$\begin{cases} (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \\ (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \end{cases}$$

are compatible; and (b) $b_8 + b_{11}x_1 = 0$, but $b_3 + b_6x_1 \neq 0$.

P28 All the leaves being $C5$, except for two of them which are $C9$. This occurs when $b_{10} = b_{13} = b_{14} = b_{15} = 0$, b_6, b_7, b_{11} and b_{12} are non null, and there are different values x_0 and x_1 such that: (a) $b_4 + b_7x_0 =$

$b_9 + b_{12}x_0 = 0$, and the systems $\begin{cases} (b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0 \\ (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \end{cases}$ and $\begin{cases} (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \\ (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \end{cases}$ are incompatible; and (b) $b_3 + b_6x_1 = b_8 + b_{11}x_1 = 0$, and the systems

$$\begin{cases} (b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0 \\ (b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0 \end{cases}$$

and

$$\begin{cases} (b_0 + b_1x_1) + X(b_4 + b_7x_1) = 0 \\ (b_2 + b_5x_1) + X(b_9 + b_{12}x_1) = 0 \end{cases}$$

are incompatible.

P29 All the leaves being $C5$, except for two of them which are $C10$. This possibility appears when $b_{10} = b_{13} = b_{14} = b_{15} = 0$, b_6, b_7, b_{11} and b_{12} are non null, and there are different values x_0 and x_1 such that: (a) $b_4 + b_7x_0 = b_9 + b_{12}x_0 = 0$, and the systems

$$\begin{cases} (b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0 \\ (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \end{cases}$$

and

$$\begin{cases} (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \\ (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \end{cases}$$

are compatible; and (b) $b_3 + b_6x_1 = b_8 + b_{11}x_1 = 0$, and the systems

$$\begin{cases} (b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0 \\ (b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0 \end{cases}$$

and

$$\begin{cases} (b_0 + b_1x_1) + X(b_4 + b_7x_1) = 0 \\ (b_2 + b_5x_1) + X(b_9 + b_{12}x_1) = 0 \end{cases}$$

are compatible.

P30 All the leaves being $C5$, except for two of them, one of which is $C9$ and the other is $C10$. This possibility emerges when $b_{10} = b_{13} = b_{14} = b_{15} = 0$, b_6, b_7, b_{11} and b_{12} are non null, and there are different values x_0 and x_1 such that: (a) $b_4 + b_7x_0 = b_9 + b_{12}x_0 = 0$, and the systems

$$\begin{cases} (b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0 \\ (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \end{cases}$$

and

$$\left\{\begin{array}{c}(b_0+b_1x_0)+X(b_3+b_6x_0)=0\\(b_2+b_5x_0)+X(b_8+b_{11}x_0)=0\end{array}\right.$$

are incompatible; and (b) $b_3+b_6x_1=b_8+b_{11}x_1=0$, and the systems

$$\left\{\begin{array}{c}(b_0+b_1x_1)+X(b_2+b_5x_1)=0\\(b_3+b_6x_1)+X(b_8+b_{11}x_1)=0\end{array}\right.$$

and

$$\left\{\begin{array}{c}(b_0+b_1x_1)+X(b_4+b_7x_1)=0\\(b_2+b_5x_1)+X(b_9+b_{12}x_1)=0\end{array}\right.$$

are compatible.

P31 All the leaves being $C5$, except for one of them which is $C1$. This case arises when $b_{10}=b_{13}=b_{14}=b_{15}=0$, b_6, b_7, b_{11} and b_{12} are non null, and there are different values x_0 such that $b_2+b_5x_0=b_3+b_6x_0=b_4+b_7x_0=b_8+b_{11}x_0=b_9+b_{12}x_0=0$, but $b_0+b_1x_0\neq 0$.

P32 All the leaves being $C5$, except for one of them which is $C11$. This possibility emerges when $b_{10}=b_{13}=b_{14}=b_{15}=0$, b_6, b_7, b_{11} and b_{12} are non null, and there are different values x_0 such that $b_0+b_1x_0=b_2+b_5x_0=b_3+b_6x_0=b_4+b_7x_0=b_8+b_{11}x_0=b_9+b_{12}x_0=0$.

Predominance of $C6$

P33 All the leaves being $C6$. This only occurs when $b_{11}=b_{12}=b_{13}=b_{14}=b_{15}=0$, and b_8, b_9 and b_{10} are non-null.

P34 All the leaves being $C6$, except for one of them which is $C5$. This possibility appears when $b_{12}=b_{13}=b_{14}=b_{15}=0$, b_9, b_{10} and b_{11} are non-null, and the system $\left\{\begin{array}{c}(b_2+b_5x_0)+Xb_9=0\\(b_3+b_6x_0)+Xb_{10}=0\end{array}\right.$ is incompatible, where x_0 is the solution of the equation $b_8+b_{11}x=0$.

P35 All the leaves being $C6$, except for two of them which are $C5$. This occurs when $b_{13}=b_{14}=b_{15}=0$, b_{10}, b_{11} and b_{12} are non-null, and the systems

$$\left\{\begin{array}{c}(b_2+b_5x_0)+X(b_9+b_{12}x_0)=0\\(b_3+b_6x_0)+Xb_{10}=0\end{array}\right.$$

and

$$\left\{\begin{array}{c}(b_2+b_5x_1)+X(b_8+b_{11}x_1)=0\\(b_4+b_7x_1)+Xb_{10}=0\end{array}\right.$$

are incompatible, where x_0 and x_1 are the solutions of the equations $b_8 + b_{11}x = 0$ and $b_9 + b_{12}x = 0$, respectively. It is required that x_0 and x_1 be different.

P36 All the leaves being $C6$, except for three of them which are $C5$. This happens when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, and the systems

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases},$$

$$\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_{10} + b_{13}x_1) = 0 \end{cases}$$

and

$$\begin{cases} (b_3 + b_6x_2) + X(b_8 + b_{11}x_2) = 0 \\ (b_4 + b_7x_2) + X(b_9 + b_{12}x_2) = 0 \end{cases}$$

are incompatible, where x_0, x_1 and x_2 are the solutions of the equations $b_8+b_{11}x = 0$, $b_9+b_{12}x = 0$ and $b_{10}+b_{13}x = 0$, respectively. It is required that x_0, x_1 and x_2 be pairwise different.

P37 All the leaves being $C6$, except for one of them which is $C3$. This possibility arises when $b_{13} = b_{14} = b_{15} = 0$, b_{10}, b_{11} and b_{12} are non-null, and the systems

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \end{cases}$$

and

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + Xb_{10} = 0 \end{cases}$$

are incompatible, where x_0 is the solution of the system $b_8 + b_{11}x = b_9 + b_{12}x = 0$.

P38 All the leaves being $C6$, except for one of them which is $C2$. This happens when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, and the systems

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases},$$

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$$

and

$$\begin{cases} (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \end{cases}$$

are incompatible, where x_0 is the solution of the system

$$\begin{cases} b_8 + b_{11}x = 0 \\ b_9 + b_{12}x = 0 \\ b_{10} + b_{13}x = 0 \end{cases}.$$

P39 All the leaves being $C6$, except for two of them, one of which is $C3$ and the other is $C5$. This happens when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, and the systems

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases},$$

$$\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_{10} + b_{13}x_1) = 0 \end{cases}$$

and

$$\begin{cases} (b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_9 + b_{12}x_1) = 0 \end{cases}$$

are incompatible, where x_0 and x_1 are the solutions of the equation $b_8 + b_{11}x = 0$ and the system $b_9 + b_{12}x = b_{10} + b_{13}x = 0$, respectively. It is required that x_0 and x_1 be different.

P40 All the leaves being $C6$, except for one of them which is $C7$. This possibility appears when $b_{12} = b_{13} = b_{14} = b_{15} = 0$, b_9, b_{10} and b_{11} are non-null, and if $b_8 + b_{11}x_0 = 0$, then the system $\begin{cases} (b_2 + b_5x_0) + Xb_9 = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$.

P41 All the leaves being $C6$, except for one of them which is $C7$. This possibility appears when $b_{12} = b_{13} = b_{14} = b_{15} = 0$, b_9, b_{10} and b_{11} are non-null, and if $b_8 + b_{11}x_0 = 0$, then the system

$$\begin{cases} (b_2 + b_5x_0) + Xb_9 = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \\ (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \end{cases}$$

is compatible.

P42 All the leaves being $C6$, except for two of them, one of which is $C5$ and the other is $C7$. This occurs when $b_{13} = b_{14} = b_{15} = 0$, b_{10}, b_{11} and b_{12} are non-null, the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, and the system $\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + Xb_{10} = 0 \end{cases}$ is incompatible, where x_0 and x_1 are the solutions of the equations $b_8 + b_{11}x = 0$ and $b_9 + b_{12}x = 0$, respectively. It is required that x_0 and x_1 be different.

P43 All the leaves being $C6$, except for two of them, one of which is $C5$ and the other is $C8$. This occurs when $b_{13} = b_{14} = b_{15} = 0$, b_{10}, b_{11} and b_{12} are non-null, the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \\ (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \end{cases}$ is compatible, and the system $\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + Xb_{10} = 0 \end{cases}$ is incompatible, where x_0 and x_1 are the solutions of the equations $b_8 + b_{11}x = 0$ and $b_9 + b_{12}x = 0$, respectively. It is required that x_0 and x_1 be different.

P44 All the leaves being $C6$, except for two of them, one of which is $C7$ and the other is $C8$. This occurs when $b_{13} = b_{14} = b_{15} = 0$, b_{10}, b_{11} and b_{12} are non-null, the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, and the system $\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + Xb_{10} = 0 \\ (b_0 + b_1x_1) + X(b_3 + b_6x_1) = 0 \end{cases}$ is compatible, where x_0 and x_1 are the solutions of the equations $b_8 + b_{11}x = 0$ and $b_9 + b_{12}x = 0$, respectively. It is required that x_0 and x_1 be different.

P45 All the leaves being $C6$, except for two of them which are $C7$. This happens when $b_{13} = b_{14} = b_{15} = 0$, b_{10}, b_{11} and b_{12} are non-null, the systems

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \end{cases}$$

and

$$\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + Xb_{10} = 0 \end{cases}$$

are compatible, but its respective solutions do not satisfy the equations $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$ and $(b_0 + b_1x_1) + X(b_3 + b_6x_1) = 0$, respectively. Here x_0 and x_1 are different values of x such that $b_8 + b_{11}x_0 = 0$ and $b_9 + b_{12}x_1 = 0$.

P46 All the leaves being $C6$, except for two of them which are $C7$. This happens when $b_{13} = b_{14} = b_{15} = 0$, b_{10}, b_{11} and b_{12} are non-null, the systems

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \end{cases}$$

and

$$\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + Xb_{10} = 0 \end{cases}$$

are compatible, but its respective solutions do not satisfy the equations $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$ and $(b_0 + b_1x_1) + X(b_3 + b_6x_1) = 0$, respectively. Here x_0 and x_1 are different values of x such that $b_8 + b_{11}x_0 = 0$ and $b_9 + b_{12}x_1 = 0$.

P47 All the leaves being $C6$, except for two of them which are $C8$. This possibility emerges when $b_{13} = b_{14} = b_{15} = 0$, b_{10}, b_{11} and b_{12} are non-null, the systems

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \\ (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \end{cases}$$

and

$$\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + Xb_{10} = 0 \\ (b_0 + b_1x_1) + X(b_3 + b_6x_1) = 0 \end{cases}$$

are compatible. Here x_0 and x_1 are different values of x such that $b_8 + b_{11}x_0 = 0$ and $b_9 + b_{12}x_1 = 0$.

P48 All the leaves being $C6$, except for three of them, two of which are $C5$ and the other is $C7$. This happens when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the systems $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$, and $\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_{10} + b_{13}x_1) = 0 \end{cases}$ are incompatible, and the system $\begin{cases} (b_3 + b_6x_2) + X(b_8 + b_{11}x_2) = 0 \\ (b_4 + b_7x_2) + X(b_9 + b_{12}x_2) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_2) + X(b_2 + b_5x_2) = 0$. Here x_0, x_1 and x_2 are the solutions of the equations $b_8 + b_{11}x = 0$, $b_9 + b_{12}x = 0$ and $b_{10} + b_{13}x = 0$, respectively. It is required that x_0, x_1 and x_2 be pairwise different.

P49 All the leaves being $C6$, except for three of them, two of which are $C5$ and the other is $C8$. This happens when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the systems $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$, and $\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_{10} + b_{13}x_1) = 0 \end{cases}$ are incompatible, and the system $\begin{cases} (b_3 + b_6x_2) + X(b_8 + b_{11}x_2) = 0 \\ (b_4 + b_7x_2) + X(b_9 + b_{12}x_2) = 0 \\ (b_0 + b_1x_2) + X(b_2 + b_5x_2) = 0 \end{cases}$ is compatible. Here x_0, x_1 and x_2 are the solutions of the equations $b_8 + b_{11}x = 0$, $b_9 + b_{12}x = 0$ and $b_{10} + b_{13}x = 0$, respectively. It is required that x_0, x_1 and x_2 be pairwise different.

P50 All the leaves being $C6$, except for three of them, two of which are $C7$ and the other is $C5$. This possibility emerges when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is incompatible, and the systems $\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_{10} + b_{13}x_1) = 0 \end{cases}$ and $\begin{cases} (b_3 + b_6x_2) + X(b_8 + b_{11}x_2) = 0 \\ (b_4 + b_7x_2) + X(b_9 + b_{12}x_2) = 0 \end{cases}$ are compatible, but its respective solution do not satisfy the equations $(b_0 + b_1x_1) + X(b_3 + b_6x_1) = 0$ and $(b_0 + b_1x_2) + X(b_2 + b_5x_2) = 0$, respectively. Here x_0, x_1 and x_2 satisfy $b_8 + b_{11}x_0 = 0$, $b_9 + b_{12}x_1 = 0$ and $b_{10} + b_{13}x_2 = 0$. It is required that x_0, x_1 and x_2 be pairwise different.

P51 All the leaves being $C6$, except for three of them, two of which are $C8$ and the other is $C5$. This possibility emerges when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is incompatible, and the systems $\begin{cases} (b_2 + b_5x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_{10} + b_{13}x_1) = 0 \\ (b_0 + b_1x_1) + X(b_3 + b_6x_1) = 0 \end{cases}$ and $\begin{cases} (b_3 + b_6x_2) + X(b_8 + b_{11}x_2) = 0 \\ (b_4 + b_7x_2) + X(b_9 + b_{12}x_2) = 0 \\ (b_0 + b_1x_2) + X(b_2 + b_5x_2) = 0 \end{cases}$ are compatible. Here x_0, x_1 and x_2 satisfy $b_8 + b_{11}x_0 = 0$, $b_9 + b_{12}x_1 = 0$ and $b_{10} + b_{13}x_2 = 0$. It is required that x_0, x_1 and x_2 be pairwise different.

P52 All the leaves being $C6$, except for three of them which are $C5$, $C7$, and $C8$, respectively. This possibility arises when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and

b_{13} are non-null, the system $\left\{\begin{array}{c}(b_2+b_5x_0)+X(b_9+b_{12}x_0)=0\\(b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0\end{array}\right.$ is incompatible, and the systems $\left\{\begin{array}{c}(b_2+b_5x_1)+X(b_8+b_{11}x_1)=0\\(b_4+b_7x_1)+X(b_{10}+b_{13}x_1)=0\\(b_0+b_1x_1)+X(b_3+b_6x_1)=0\end{array}\right.$ and $\left\{\begin{array}{c}(b_3+b_6x_2)+X(b_8+b_{11}x_2)=0\\(b_4+b_7x_2)+X(b_9+b_{12}x_2)=0\end{array}\right.$ are compatible, but the solution of the latter does not satisfy the equation $(b_0+b_1x_2)+X(b_2+b_5x_2)=0$. Here x_0, x_1 and x_2 satisfy $b_8+b_{11}x_0=0$, $b_9+b_{12}x_1=0$ and $b_{10}+b_{13}x_2=0$. It is required that x_0, x_1 and x_2 be pairwise different.

P53 All the leaves being $C6$, except for three of them which are $C7$. This possibility arises when $b_{14}=b_{15}=0$, b_{11}, b_{12} and b_{13} are non-null, the systems

$$\left\{\begin{array}{c}(b_2+b_5x_0)+X(b_9+b_{12}x_0)=0\\(b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0\end{array}\right.,$$

$$\left\{\begin{array}{c}(b_2+b_5x_1)+X(b_8+b_{11}x_1)=0\\(b_4+b_7x_1)+X(b_{10}+b_{13}x_1)=0\end{array}\right.$$

and

$$\left\{\begin{array}{c}(b_3+b_6x_2)+X(b_8+b_{11}x_2)=0\\(b_4+b_7x_2)+X(b_9+b_{12}x_2)=0\end{array}\right.$$

are compatible, but its respective solutions fail to satisfy the equations $(b_0+b_1x_1)+X(b_4+b_7x_1)=0$, $(b_0+b_1x_1)+X(b_3+b_6x_1)=0$, and $(b_0+b_1x_2)+X(b_2+b_5x_2)=0$, respectively. Here x_0, x_1 and x_2 satisfy $b_8+b_{11}x_0=0$, $b_9+b_{12}x_1=0$ and $b_{10}+b_{13}x_2=0$. It is required that x_0, x_1 and x_2 be pairwise different.

P54 All the leaves being $C6$, except for three of them which are $C8$. This occurs when $b_{14}=b_{15}=0$, b_{11}, b_{12} and b_{13} are non-null, the systems

$$\left\{\begin{array}{c}(b_2+b_5x_0)+X(b_9+b_{12}x_0)=0\\(b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0\\(b_0+b_1x_1)+X(b_4+b_7x_1)=0\end{array}\right.,$$

$$\left\{\begin{array}{c}(b_2+b_5x_1)+X(b_8+b_{11}x_1)=0\\(b_4+b_7x_1)+X(b_{10}+b_{13}x_1)=0\\(b_0+b_1x_1)+X(b_3+b_6x_1)=0\end{array}\right.$$

and

$$\left\{\begin{array}{c}(b_3+b_6x_2)+X(b_8+b_{11}x_2)=0\\(b_4+b_7x_2)+X(b_9+b_{12}x_2)=0\\(b_0+b_1x_2)+X(b_2+b_5x_2)=0\end{array}\right.$$

are compatible. Here x_0, x_1 and x_2 satisfy $b_8+b_{11}x_0=0$, $b_9+b_{12}x_1=0$ and $b_{10}+b_{13}x_2=0$. It is required that x_0, x_1 and x_2 be pairwise different.

P55 All the leaves being $C6$, except for three of them, two of which are $C7$ and the other is $C8$. This possibility appears when $b_{14}=b_{15}=0$, b_{11}, b_{12} and b_{13} are non-null, the systems $\begin{cases}(b_2+b_5x_0)+X(b_9+b_{12}x_0)=0\\(b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0\end{cases}$ and $\begin{cases}(b_2+b_5x_1)+X(b_8+b_{11}x_1)=0\\(b_4+b_7x_1)+X(b_{10}+b_{13}x_1)=0\end{cases}$ are compatible, but its respective solutions do not satisfy the equations $(b_0+b_1x_1)+X(b_4+b_7x_1)=0$ and $(b_0+b_1x_1)+X(b_3+b_6x_1)=0$, respectively, and the system $\begin{cases}(b_3+b_6x_2)+X(b_8+b_{11}x_2)=0\\(b_4+b_7x_2)+X(b_9+b_{12}x_2)=0\\(b_0+b_1x_2)+X(b_2+b_5x_2)=0\end{cases}$ is compatible. Here x_0, x_1 and x_2 satisfy $b_8+b_{11}x_0=0$, $b_9+b_{12}x_1=0$ and $b_{10}+b_{13}x_2=0$. It is required that x_0, x_1 and x_2 be pairwise different.

P56 All the leaves being $C6$, except for three of them, two of which are $C8$ and the other is $C7$. This occurs when $b_{14}=b_{15}=0$, b_{11}, b_{12} and b_{13} are non-null, the systems $\begin{cases}(b_2+b_5x_0)+X(b_9+b_{12}x_0)=0\\(b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0\\(b_0+b_1x_1)+X(b_4+b_7x_1)=0\end{cases}$ and

$$\begin{cases}(b_2+b_5x_1)+X(b_8+b_{11}x_1)=0\\(b_4+b_7x_1)+X(b_{10}+b_{13}x_1)=0\\(b_0+b_1x_1)+X(b_3+b_6x_1)=0\end{cases},$$

and

$$\begin{cases}(b_3+b_6x_2)+X(b_8+b_{11}x_2)=0\\(b_4+b_7x_2)+X(b_9+b_{12}x_2)=0\end{cases}$$

are compatible, but the solution of the latter fails to satisfy the equation $(b_0+b_1x_2)+X(b_2+b_5x_2)=0$. Here x_0, x_1 and x_2 satisfy $b_8+b_{11}x_0=0$, $b_9+b_{12}x_1=0$ and $b_{10}+b_{13}x_2=0$. It is required that x_0, x_1 and x_2 be pairwise different.

P57 All the leaves being $C6$, except for one of them which is $C9$. This occurs when $b_{13}=b_{14}=b_{15}=0$, b_{10}, b_{11} and b_{12} are non-null, and the systems

$$\begin{cases}(b_2+b_5x_0)+X(b_9+b_{12}x_0)=0\\(b_3+b_6x_0)+Xb_{10}=0\end{cases}$$

and
$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_)) + Xb_{10} = 0 \end{cases}$$
are compatible, but its respective solutions fail to satisfy equations $(b_0+b_1x_0)+X(b_4+b_7x_0) = 0$ and $(b_0+b_1x_0)+X(b_3+b_6x_0) = 0$, respectively. Here x_0 is the solutions of the system $b_8 + b_{11}x = b_9 + b_{12}x = 0$.

P58 All the leaves being $C6$, except for one of them which is $C10$. This occurs when $b_{13} = b_{14} = b_{15} = 0$, b_{10}, b_{11} and b_{12} are non-null, and the systems
$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \\ (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \end{cases}$$
and
$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_)) + Xb_{10} = 0 \\ (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \end{cases}$$
are compatible, where x_0 is the solutions of the system $b_8 + b_{11}x = b_9 + b_{12}x = 0$.

P59 All the leaves being $C6$, except for one of them which is $C1$. This only occurs when $b_{14} = b_{15} = 0$, b_{11}, b_{12}, and b_{13} are non-null, and there exists x_0 such that $b_2 + b_5x_0 = b_3 + b_6x_0 = b_4 + b_7x_0 = b_5 + b_9x_0 = b_8 + b_{11}x_0 = b_9 + b_{12}x_0 = b_{10} + b_{13}x_0 = 0$, but $b_0 + b_1x_0 \neq 0$.

P60 All the leaves being $C6$, except for one of them which is $C11$. This only occurs when $b_{14} = b_{15} = 0$, b_{11}, b_{12}, and b_{13} are non-null, and there exists x_0 such that $b_0 + b_1x_0 = b_2 + b_5x_0 = b_3 + b_6x_0 = b_4 + b_7x_0 = b_5 + b_9x_0 = b_8 + b_{11}x_0 = b_9 + b_{12}x_0 = b_{10} + b_{13}x_0 = 0$.

61 All the leaves being $C6$, except for two of them, one of which is $C3$ and the other $C7$. This happens when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the systems $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ and $\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ are incompatible, and the system $\begin{cases} (b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_9 + b_{12}x_1) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0$, where x_0 and x_1 are the solutions of the systems $b_8 + b_{11}x = b_9 + b_{12}x = 0$ and $b_{10} + b_{13}x = 0$, respectively. It is required that $x_0 \neq x_1$.

62 All the leaves being $C6$, except for two of them, one of which is $C3$ and the other is $C8$. This possibility appears when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the systems $\left\{\begin{array}{c}(b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0\\(b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0\end{array}\right.$ and $\left\{\begin{array}{c}(b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0\\(b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0\end{array}\right.$ are incompatible, and the system $\left\{\begin{array}{c}(b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0\\(b_4 + b_7x_1) + X(b_9 + b_{12}x_1) = 0\\(b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0\end{array}\right.$ is compatible, where x_0 and x_1 are the solutions of the systems $b_8+b_{11}x = b_9+b_{12}x = 0$ and $b_{10}+b_{13}x = 0$, respectively. It is required that $x_0 \neq x_1$.

63 All the leaves being $C6$, except for two of them, one of which is $C5$ and the other is $C7$. This possibility emerges when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the systems $\left\{\begin{array}{c}(b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0\\(b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0\end{array}\right.$ and $\left\{\begin{array}{c}(b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0\\(b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0\end{array}\right.$ are compatible, but its respective solutions fail to satisfy the equations $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$ and $(b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0$, respectively, and the system $\left\{\begin{array}{c}(b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0\\(b_4 + b_7x_1) + X(b_9 + b_{12}x_1) = 0\end{array}\right.$ is incompatible, where x_0 and x_1 are the solutions of the systems $b_8+b_{11}x = b_9+b_{12}x = 0$ and $b_{10}+b_{13}x = 0$, respectively. It is required that $x_0 \neq x_1$.

64 All the leaves being $C6$, except for two of them, one of which is $C5$ and the other is $C8$. This occurs when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the systems $\left\{\begin{array}{c}(b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0\\(b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0\\(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0\end{array}\right.$ and $\left\{\begin{array}{c}(b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0\\(b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0\\(b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0\end{array}\right.$ are compatible, and the system $\left\{\begin{array}{c}(b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0\\(b_4 + b_7x_1) + X(b_9 + b_{12}x_1) = 0\end{array}\right.$ is incompatible, where x_0 and x_1 are the solutions of the systems $b_8+b_{11}x = b_9+b_{12}x = 0$ and $b_{10}+b_{13}x = 0$, respectively. It is required that $x_0 \neq x_1$.

65 All the leaves being $C6$, except for two of them, one of which is $C8$ and the other is $C9$. This happens when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the systems

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases},$$

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$$

and

$$\begin{cases} (b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_9 + b_{12}x_1) = 0 \\ (b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0 \end{cases}$$

are compatible, but the solutions of the two first do not satisfy the equations $(b_0+b_1x_0)+X(b_4+b_7x_0) = 0$ and $(b_0+b_1x_0)+X(b_3+b_6x_0) = 0$, respectively, where x_0 and x_1 are the solutions of the systems $b_8+b_{11}x = b_9+b_{12}x = 0$ and $b_{10}+b_{13}x = 0$, respectively. It is required that $x_0 \neq x_1$.

66 All the leaves being $C6$, except for two of them, one of which is $C7$ and the other is $C10$. This occurs when $b_{14} = b_{15} = 0$, b_{11}, b_{12} and b_{13} are non-null, the systems $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \end{cases}$ and

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \end{cases}$$

and

$$\begin{cases} (b_3 + b_6x_1) + X(b_8 + b_{11}x_1) = 0 \\ (b_4 + b_7x_1) + X(b_9 + b_{12}x_1) = 0 \end{cases}$$

are compatible, but the solution of the latter fails to satisfy the equation $(b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0$, where x_0 and x_1 are the solutions of the systems $b_8 + b_{11}x = b_9 + b_{12}x = 0$ and $b_{10} + b_{13}x = 0$, respectively. It is required that $x_0 \neq x_1$.

Predominance of $C7$

67 All the leaves being $C7$. This possibility appears when $b_8 = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, b_9 and b_{10} are non-null, and for all value x_0 of x the system $\begin{cases} (b_2 + b_5x_0) + Xb_9 = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$.

68 All the leaves being $C7$, except for one of them which is $C3$. This possibility arises when $b_8 = b_{11} = b_{12} = b_{14} = b_{15} = 0$, b_9, b_{10} and b_{13} are non-null, for all value x_0 of x, except for a value x_1, the system $\begin{cases} (b_2 + b_5x_0) + Xb_9 = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, and $b_{10} + b_{13}x_1 = 0$ and $b_3 + b_6x_1 \neq 0$.

69 All the leaves being $C7$, except for two of them which are $C3$. This possibility arises when $b_8 = b_{11} = b_{14} = b_{15} = 0$, b_9, b_{10}, b_{12} and b_{13} are non-null, for all value x_0 of x, except for two distinct values x_1 and x_2, the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, and $b_9 + b_{12}x_2 = b_{10} + b_{13}x_1 = 0$ and $b_3 + b_6x_1$ and $b_4 + b_7x_2$ are non-null.

70 All the leaves being $C7$, except for one of them which is $C2$. This possibility arises when $b_8 = b_{11} = b_{14} = b_{15} = 0$, b_9, b_{10}, b_{12} and b_{13} are non-null, for all value x_0 of x, except for one value x_1, the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, and $b_9 + b_{12}x_1 = b_{10} + b_{13}x_1 = 0$ and $b_3 + b_6x_1$ and $b_4 + b_7x_1$ are non-null.

71 All the leaves being $C7$, except for one of them which is $C8$. This possibility emerges when $b_8 = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, b_9 and b_{10} are non-null, and for all value x_0 of x the system $\begin{cases} (b_2 + b_5x_0) + Xb_9 = 0 \\ (b_3 + b_6x_0) + Xb_{10} = 0 \end{cases}$ is compatible, but only for one value, the solution of system satisfies the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$.

72 All the leaves being $C7$, except for one of them which is $C9$. This happens when $b_8 = b_{11} = b_{12} = b_{14} = b_{15} = 0$, b_9, b_{10} and b_{13} are non-null, for all value x_0 of x the system $\begin{cases} (b_2 + b_5x_0) + Xb_9 = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, and there exists a value x_1 such that $b_3 + b_6x_1 = b_{10} + b_{13}x_1 = 0$, and the system $\begin{cases} (b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0 \\ (b_4 + b_7x_1) + Xb_9 = 0 \end{cases}$ is incompatible.

73 All the leaves being $C7$, except for one of them which is $C10$. This occurs when $b_8 = b_{11} = b_{12} = b_{14} = b_{15} = 0$, b_9, b_{10} and b_{13} are non-null, for all value x_0 of x the system $\begin{cases} (b_2 + b_5x_0) + Xb_9 = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, and there exists a value x_1 such that $b_3 + b_6x_1 = b_{10} + b_{13}x_1 = 0$, and the system $\begin{cases} (b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0 \\ (b_4 + b_7x_1) + Xb_9 = 0 \end{cases}$ is compatible.

74 All the leaves being $C7$, except for one of them which is $C4$. This occurs when $b_8 = b_{11} = b_{14} = b_{15} = 0$, b_9 and b_{10} are non-null, for all value x_0 of x the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0+b_1x_0)+X(b_4+b_7x_0) = 0$, and there exists a value x_1 such that $b_2+b_5x_1 = b_3+b_6x_1 = b_9+b_{12}x_1 = b_{10} + b_{13}x_1 = 0$ and $b_4 + b_7x_1 \neq 0$.

75 All the leaves being $C7$, except for one of them which is $C1$. This possibility arises when $b_8 = b_{11} = b_{14} = b_{15} = 0$, b_9 and b_{10} are non-null, for all value x_0 of x the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, and there exists a value x_1 such that $b_2 + b_5x_1 = b_3 + b_6x_1 = b_4 + b_7x_1 = b_9 + b_{12}x_1 = b_{10} + b_{13}x_1 = 0$ and $b_0 + b_1x_1 \neq 0$.

76 All the leaves being $C7$, except for one of them which is $C11$. This possibility arises when $b_8 = b_{11} = b_{14} = b_{15} = 0$, b_9 and b_{10} are non-null, for all value x_0 of x the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, and there exists a value x_1 such that $b_0 + b_1x_1 = b_2 + b_5x_1 = b_3 + b_6x_1 = b_4 + b_7x_1 = b_9 + b_{12}x_1 = b_{10} + b_{13}x_1 = 0$.

Predominance of $C8$

77 All the leaves being $C8$. This possibility appears when $b_8 = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, b_9 and b_{10} are non-null, and for all value x_0 of x the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \end{cases}$ is compatible.

78 All the leaves being $C8$, except for one of them which is $C10$. This possibility emerges when $b_8 = b_{11} = b_{12} = b_{14} = b_{15} = 0$, b_9, b_{10} and b_{13}

are non-null, for all value x_0 of x the system

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \end{cases}$$

is compatible, and there exists a value x_1 such that $b_3 + b_6x_1 = b_{10} + b_{13}x_1 = 0$ and the system $\begin{cases} (b_0 + b_1x_1) + X(b_2 + b_5x_1) = 0 \\ (b_4 + b_7x_1) + X(b_9 + b_{12}x_0) = 0 \end{cases}$ is compatible.

79 All the leaves being $C8$, except for one of them which is $C11$. This possibility arises when $b_8 = b_{11} = b_{14} = b_{15} = 0$, b_9, b_{10}, b_{12} and b_{13} are non-null, for all value x_0 of x the system $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \end{cases}$ is compatible, and there exists a value x_1 such that $b_0 + b_1x_1 = b_2 + b_5x_1 = b_3 + b_6x_1 = b_4 + b_7x_1 = b_9 + b_{12}x_1 = b_{10} + b_{13}x_1 = 0$.

80 All the leaves being $C9$. This possibility appears when $b_2 = b_5 = b_8 = b_9 = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = 0$, $b_{10} \neq 0$, and for all value x_0 of x the system $\begin{cases} (b_4 + b_7x_0) + X(b_{10}) = 0 \\ (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \end{cases}$ is incompatible.

Predominance of $C9$

81 All the leaves being $C9$, except for one which is $C2$. This possibility appears when $b_2 = b_5 = b_8 = b_9 = b_{11} = b_{12} = b_{14} = b_{15} = 0$, $b_{10} \neq 0$, and for all value x_0 of x the system $\begin{cases} (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \end{cases}$ is incompatible, except for a value x_1 such that $b_{10} + b_{13}x_1 = 0$ and $b_3 + b_6x_1$ and $b_4 + b_7x_1$ are non-null.

82 All the leaves being $C9$, except for one which is $C4$. This happens when $b_2 = b_5 = b_8 = b_9 = b_{11} = b_{12} = b_{14} = b_{15} = 0$, $b_{10} \neq 0$, and for all value x_0 of x the system $\begin{cases} (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \end{cases}$ is incompatible, except for a value x_1 such that $b_{10} + b_{13}x_1 = 0$ and only one of the numbers $b_3 + b_6x_1$ and $b_4 + b_7x_1$ is non-null.

83 All the leaves being $C9$, except for one which is $C4$. This occurs when $b_2 = b_5 = b_8 = b_9 = b_{11} = b_{12} = b_{14} = b_{15} = 0$, $b_{10} \neq 0$, and for all value

x_0 of x the system $\begin{cases} (b_4+b_7x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_0+b_1x_0)+X(b_3+b_6x_0)=0 \end{cases}$ is incompatible, except for a value x_1 such that $b_3+b_6x_1=b_4+b_7x_1=b_{10}+b_{13}x_1=0$ and $b_0+b_1x_1$ is non-null.

84 All the leaves being $C9$, except for one which is $C11$. This occurs when $b_2=b_5=b_8=b_9=b_{11}=b_{12}=b_{14}=b_{15}=0$, $b_{10}\neq 0$, and for all value x_0 of x the system $\begin{cases} (b_4+b_7x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_0+b_1x_0)+X(b_3+b_6x_0)=0 \end{cases}$ is incompatible, except for a value x_1 such that $b_0+b_1x_1=b_3+b_6x_1=b_4+b_7x_1=b_{10}+b_{13}x_1=0$.

Predominance of $C10$

85 All the leaves being $C10$. This happens when $b_2=b_5=b_8=b_9=b_{11}=b_{12}=b_{13}=b_{14}=b_{15}=0$, $b_{10}\neq 0$, and for all value x_0 of x the system $\begin{cases} (b_4+b_7x_0)+Xb_{10}=0 \\ (b_0+b_1x_0)+X(b_3+b_6x_0)=0 \end{cases}$ is compatible.

86 All the leaves being $C10$, except for one of them which is $C2$. This possibility emerges when $b_2=b_5=b_8=b_9=b_{11}=b_{12}=b_{14}=b_{15}=0$, $b_{10},b_{13}\neq 0$, for all value x_0 of x, except for a value x_1, the system $\begin{cases} (b_4+b_7x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_0+b_1x_0)+X(b_3+b_6x_0)=0 \end{cases}$ is compatible, and $b_{10}+b_{13}x_1=0$ and $b_3+b_6x_1$ and $b_4+b_7x_1$ are non-null.

87 All the leaves being $C10$, except for one of them which is $C1$. This possibility arises when $b_2=b_5=b_8=b_9=b_{11}=b_{12}=b_{14}=b_{15}=0$, $b_{10},b_{13}\neq 0$, for all value x_0 of x, except for a value x_1, the system $\begin{cases} (b_4+b_7x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_0+b_1x_0)+X(b_3+b_6x_0)=0 \end{cases}$ is compatible, and $b_3+b_6x_1=b_4+b_7x_1=b_{10}+b_{13}x_1=0$ and $b_0+b_1x_1$ is non-null.

88 All the leaves being $C10$, except for one of them which is $C11$. This possibility arises when $b_2=b_5=b_8=b_9=b_{11}=b_{12}=b_{14}=b_{15}=0$, $b_{10},b_{13}\neq 0$, for all value x_0 of x, except for a value x_1, the system $\begin{cases} (b_4+b_7x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_0+b_1x_0)+X(b_3+b_6x_0)=0 \end{cases}$ is compatible, and $b_0+b_1x_1=b_3+b_6x_1=b_4+b_7x_1=b_{10}+b_{13}x_1=0$.

Predominance of $C11$

89 All the leaves being $C11$. This only occurs when all coefficients are null.

Predominance of $C12$

90 All the leaves being $C12$. This possibility emerges when $b_{15} = 0$, $b_{14} \neq 0$, and for all value x_0 of x, the systems

$$\begin{cases} (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_{10} + b_{13}x_0) + Xb_{14} = 0 \end{cases},$$

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + Xb_{14} = 0 \end{cases},$$

and

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + Xb_{14} = 0 \end{cases}$$

are compatible, but its respective solutions do not satisfy the equations $(b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0$, $(b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0$, and $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, respectively.

91 All the leaves being $C12$, except for one of them which is $C1$. This possibility emerges when $b_{15} \neq 0$, for all value x_0 of x, the systems

$$\begin{cases} (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_{10} + b_{13}x_0) + X(b_{14} + b_{15}) = 0 \end{cases},$$

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + X(b_{14} + b_{15}) = 0 \end{cases},$$

and

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$$

are compatible, but its respective solutions do not satisfy the equations $(b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0$, $(b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0$, and $(b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0$, respectively, and there exists a value x_1 such that $b_{14} + b_{15}x_1 = 0$ and $b_0 + b_1x_1 \neq 0$.

Predominance of $C13$

92 All the leaves being $C13$. This possibility arises when $b_{15} = 0$, $b_{14} \neq 0$, and for all value x_0 of x, the system $\begin{cases} (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_{10} + b_{13}x_0) + Xb_{14} = 0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0+b_1x_0)+X(b_2+b_5x_0) = 0$, and the systems $\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + Xb_{14} = 0 \end{cases}$, and $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + Xb_{14} = 0 \end{cases}$ are incompatible.

93 All the leaves being $C13$, except for one of them which is $C1$. This happens when $b_{15} \neq 0$, for all value x_0 of x, the system

$$\begin{cases} (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_{10} + b_{13}x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$$

is compatible, but its solution does not satisfy the equation $(b_0+b_1x_0)+X(b_2+b_5x_0) = 0$, and the systems $\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$, and $\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$ are incompatible, and there exists a value x_1 such that $b_2+b_5x_1 = b_3+b_6x_1 = b_4+b_7x_1 = b_8+b_{11}x_1 = b_9 + b_{12}x_1 = b_{10} + b_{13}x_1 = b_{14} + b_{15}x_1 = 0$, but $b_0 + b_1x_1 \neq 0$.

94 All the leaves being $C13$, except for one of them which is $C4$. This occurs when $b_{15} \neq 0$, for all value x_0 of x, the system

$$\begin{cases} (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_{10} + b_{13}x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$$

is compatible, but its solution does not satisfy the equation $(b_0+b_1x_0)+X(b_2+b_5x_0) = 0$, and the systems $\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$, and $\begin{cases} (b_2+b_5x_0)+X(b_9+b_{12}x_0)=0 \\ (b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_8+b_{11}x_0)+X(b_{14}+b_{15})=0 \end{cases}$ are incompatible, and there exists a value x_1 such that $b_3+b_6x_1 = b_4+b_7x_1 = b_8+b_{11}x_1 = b_9+b_{12}x_1 = b_{10}+b_{13}x_1 = b_{14}+b_{15}x_1 = 0$, but $b_0+b_1x_1$ and $b_2+b_5x_1$ are non-null.

95 All the leaves being $C13$, except for one of them which is $C7$. This occurs when $b_{15}\neq 0$, for all value x_0 of x, the system

$$\begin{cases} (b_3+b_6x_0)+X(b_8+b_{11}x_0)=0 \\ (b_4+b_7x_0)+X(b_9+b_{12}x_0)=0 \\ (b_{10}+b_{13}x_0)+X(b_{14}+b_{15})=0 \end{cases}$$

is compatible, but its solution does not satisfy the equation $(b_0+b_1x_0)+X(b_2+b_5x_0)=0$, and the systems $\begin{cases} (b_2+b_5x_0)+X(b_8+b_{11}x_0)=0 \\ (b_4+b_7x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_9+b_{12}x_0)+X(b_{14}+b_{15})=0 \end{cases}$, and $\begin{cases} (b_2+b_5x_0)+X(b_9+b_{12}x_0)=0 \\ (b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_8+b_{11}x_0)+X(b_{14}+b_{15})=0 \end{cases}$ are incompatible, and there exists a value x_1 such that $b_{10}+b_{13}x_1 = b_{14}+b_{15}x_1 = 0$, but $b_8+b_{11}x_1$ and $b_9+b_{12}x_1$ are non-null.

96 All the leaves being $C13$, except for one of them which is $C3$. This possibility appears when $b_{15}\neq 0$, for all value x_0 of x, the system $\begin{cases} (b_3+b_6x_0)+X(b_8+b_{11}x_0)=0 \\ (b_4+b_7x_0)+X(b_9+b_{12}x_0)=0 \\ (b_{10}+b_{13}x_0)+X(b_{14}+b_{15})=0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0+b_1x_0)+X(b_2+b_5x_0)=0$, and the systems

$$\begin{cases} (b_2+b_5x_0)+X(b_8+b_{11}x_0)=0 \\ (b_4+b_7x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_9+b_{12}x_0)+X(b_{14}+b_{15})=0 \end{cases},$$

and

$$\begin{cases} (b_2+b_5x_0)+X(b_9+b_{12}x_0)=0 \\ (b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_8+b_{11}x_0)+X(b_{14}+b_{15})=0 \end{cases}$$

are incompatible, and there exists a value x_1 such that $b_8+b_{11}x_1 = b_9+b_{12}x_1 = b_{14}+b_{15}x_1 = 0$, and $b_{10}+b_{13}x_1$ and $b_2+b_5x_1$ are non-null.

97 All the leaves being $C13$, except for one of them which is $C3$. This possibility appears when $b_{15} \neq 0$, for all value x_0 of x, the system $\begin{cases} (b_3+b_6x_0)+X(b_8+b_{11}x_0)=0 \\ (b_4+b_7x_0)+X(b_9+b_{12}x_0)=0 \\ (b_{10}+b_{13}x_0)+X(b_{14}+b_{15})=0 \end{cases}$ is compatible, but its solution does not satisfy the equation $(b_0+b_1x_0)+X(b_2+b_5x_0)=0$, and the systems

$$\begin{cases} (b_2+b_5x_0)+X(b_8+b_{11}x_0)=0 \\ (b_4+b_7x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_9+b_{12}x_0)+X(b_{14}+b_{15})=0 \end{cases},$$

and

$$\begin{cases} (b_2+b_5x_0)+X(b_9+b_{12}x_0)=0 \\ (b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_8+b_{11}x_0)+X(b_{14}+b_{15})=0 \end{cases}$$

are incompatible, and there exists a value x_1 such that $b_2+b_5x_1 = b_8+b_{11}x_1 = b_9+b_{12}x_1 = b_{14}+b_{15}x_1 = 0$, $b_{10}+b_{13}x_1$ is non-null, and the system $\begin{cases} (b_0+b_1x_1)+X(b_3+b_6x_1)=0 \\ (b_4+b_7x_1)+X(b_{10}+b_{13}x_1)=0 \end{cases}$ is incompatible.

Predominance of $C14$

98 All the leaves being $C14$. This happens when $b_{15}=0$, $b_{14}\neq 0$, and for all value x_0 of x, the systems

$$\begin{cases} (b_3+b_6x_0)+X(b_8+b_{11}x_0)=0 \\ (b_4+b_7x_0)+X(b_9+b_{12}x_0)=0 \\ (b_{10}+b_{13}x_0)+Xb_{14}=0 \end{cases},$$

$$\begin{cases} (b_2+b_5x_0)+X(b_8+b_{11}x_0)=0 \\ (b_4+b_7x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_9+b_{12}x_0)+Xb_{14}=0 \end{cases},$$

and

$$\begin{cases} (b_2+b_5x_0)+X(b_9+b_{12}x_0)=0 \\ (b_3+b_6x_0)+X(b_{10}+b_{13}x_0)=0 \\ (b_8+b_{11}x_0)+Xb_{14}=0 \end{cases}$$

are incompatible

99 All the leaves being $C14$, except for one of them which is $C6$. This occurs when $b_{15}\neq 0$, for all value x_0 of x, the systems

$$\begin{cases} (b_3+b_6x_0)+X(b_8+b_{11}x_0)=0 \\ (b_4+b_7x_0)+X(b_9+b_{12}x_0)=0 \\ (b_{10}+b_{13}x_0)+X(b_{14}+x_0b_{15})=0 \end{cases},$$

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + X(b_{14} + x_0b_{15}) = 0 \end{cases},$$

and

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + X(b_{14} + x_0b_{15}) = 0 \end{cases}$$

are incompatible, and there exists a value x_1 such that $b_{14} + x_1b_{15} = 0$, but $b_8 + x_1b_{11}$, $b_9 + x_1b_{12}$, and $b_{10} + x_1b_{13}$ are non-null.

100 All the leaves being $C14$, except for one of them which is $C5$. This possibility appears when $b_{15} \neq 0$, for all value x_0 of x, the systems

$$\begin{cases} (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_{10} + b_{13}x_0) + X(b_{14} + x_0b_{15}) = 0 \end{cases},$$

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + X(b_{14} + x_0b_{15}) = 0 \end{cases},$$

and

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + X(b_{14} + x_0b_{15}) = 0 \end{cases}$$

are incompatible, and there exists a value x_1 such that $b_8 + x_1b_{11} = b_9 + x_1b_{12} = b_{14} + x_1b_{15} = 0$, $b_{10} + x_1b_{13}$ is non-null, and the systems obtained by removing the thirst equation in the two latter systems above are incompatible.

101 All the leaves being $C14$, except for one of them which is $C2$. This possibility emerges when $b_{15} \neq 0$, for all value x_0 of x, the systems

$$\begin{cases} (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_{10} + b_{13}x_0) + X(b_{14} + x_0b_{15}) = 0 \end{cases},$$

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + X(b_{14} + x_0b_{15}) = 0 \end{cases},$$

and

$$\begin{cases} (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + X(b_{14} + x_0b_{15}) = 0 \end{cases}$$

are incompatible, and there exists a value x_1 such that $b_8 + x_1 b_{11} = b_9 + x_1 b_{12} = b_{10} + x_1 b_{13} = b_{14} + x_1 b_{15} = 0$, and the systems obtained by removing the thirst equation in the systems above are incompatible.

Predominance of $C15$

102 All the leaves being $C15$. This possibility appears when $b_{15} = 0$, $b_{14} \neq 0$, for all value x_0 of x, the systems $\begin{cases} (b_3 + b_6 x_0) + X(b_8 + b_{11} x_0) = 0 \\ (b_4 + b_7 x_0) + X(b_9 + b_{12} x_0) = 0 \\ (b_{10} + b_{13} x_0) + X b_{14} = 0 \end{cases}$ and $\begin{cases} (b_2 + b_5 x_0) + X(b_8 + b_{11} x_0) = 0 \\ (b_4 + b_7 x_0) + X(b_{10} + b_{13} x_0) = 0 \\ (b_9 + b_{12} x_0) + X b_{14} = 0 \end{cases}$ are incompatible, and the system $\begin{cases} (b_0 + b_1 x_0) + X(b_4 + b_7 x_0) = 0 \\ (b_2 + b_5 x_0) + X(b_9 + b_{12} x_0) = 0 \\ (b_3 + b_6 x_0) + X(b_{10} + b_{13} x_0) = 0 \\ (b_8 + b_{11} x_0) + X b_{14} = 0 \end{cases}$ is compatible.

103 All the leaves being $C15$, except for one of them which is $C8$. This occurs when $b_{15} \neq 0$, for all value x_0 of x, the systems

$$\begin{cases} (b_3 + b_6 x_0) + X(b_8 + b_{11} x_0) = 0 \\ (b_4 + b_7 x_0) + X(b_9 + b_{12} x_0) = 0 \\ (b_{10} + b_{13} x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$$

and

$$\begin{cases} (b_2 + b_5 x_0) + X(b_8 + b_{11} x_0) = 0 \\ (b_4 + b_7 x_0) + X(b_{10} + b_{13} x_0) = 0 \\ (b_9 + b_{12} x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$$

are incompatible, the system $\begin{cases} (b_0 + b_1 x_0) + X(b_4 + b_7 x_0) = 0 \\ (b_2 + b_5 x_0) + X(b_9 + b_{12} x_0) = 0 \\ (b_3 + b_6 x_0) + X(b_{10} + b_{13} x_0) = 0 \\ (b_8 + b_{11} x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$ is compatible, and there exists a value x_1 such that $b_8 + b_{11} x_1 = b_{14} + b_{15} x_1 = 0$, but $b_9 + b_{12} x_1$ and $b_{10} + b_{13} x_1$ are non-null.

104 All the leaves being $C15$, except for one of them which is $C3$. This occurs when $b_{15} \neq 0$, for all value x_0 of x, the systems

$$\begin{cases} (b_3 + b_6 x_0) + X(b_8 + b_{11} x_0) = 0 \\ (b_4 + b_7 x_0) + X(b_9 + b_{12} x_0) = 0 \\ (b_{10} + b_{13} x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$$

and

$$\begin{cases} (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$$

are incompatible, the system $\begin{cases} (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \\ (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + X(b_{14} + b_{15}) = 0 \end{cases}$ is compatible, and there exists a value x_1 such that $b_9 + b_{12}x_1 = b_{10} + b_{13}x_1 = b_{14} + b_{15}x_1 = 0$, but $b_8 + b_{11}x_1$ is non-null.

Predominance of $C16$

105 All the leaves being $C16$. This possibility appears when $b_{15} = 0$, $b_{14} \neq 0$, and for all value x_0 of x, the systems

$$\begin{cases} (b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0 \\ (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_{10} + b_{13}x_0) + Xb_{14} = 0 \end{cases}$$

and

$$\begin{cases} (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \\ (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + Xb_{14} = 0 \end{cases},$$

and

$$\begin{cases} (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \\ (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + Xb_{14} = 0 \end{cases}$$

are compatible.

106 All the leaves being $C16$. This possibility appears when $b_{15} = 0$, $b_{14} \neq 0$, for all value x_0 of x, the systems

$$\begin{cases} (b_0 + b_1x_0) + X(b_2 + b_5x_0) = 0 \\ (b_3 + b_6x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_{10} + b_{13}x_0) + X(b_{14} + b_{15}x_0) = 0 \end{cases}$$

and

$$\begin{cases} (b_0 + b_1x_0) + X(b_3 + b_6x_0) = 0 \\ (b_2 + b_5x_0) + X(b_8 + b_{11}x_0) = 0 \\ (b_4 + b_7x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_9 + b_{12}x_0) + X(b_{14} + b_{15}x_0) = 0 \end{cases},$$

and

$$\begin{cases} (b_0 + b_1x_0) + X(b_4 + b_7x_0) = 0 \\ (b_2 + b_5x_0) + X(b_9 + b_{12}x_0) = 0 \\ (b_3 + b_6x_0) + X(b_{10} + b_{13}x_0) = 0 \\ (b_8 + b_{11}x_0) + X(b_{14} + b_{15}x_0) = 0 \end{cases}$$

are compatible, and there exists a value x_1 such that $b_{14} + b_{15}x_1 = 0$.

Bibliography

[1] W. E. Lorensen and H. E. Cline, *Marching Cubes: a high resolution 3D surface construction algorithm,* In Siggraph **21**, 163-169. ACM, 1987.

[2] G. Nielson, *On marching cubes* IEEE Trans. on Visualiz. **9** (3), 283-297, 2003.

[3] T. Lewiner, H. Lopes, A. Vieira and G. Tavares, *Efficient implementation of marching cubes' cases with topological guarantiees* J. of Graphics Tools **8** (2), 1-16, 2003.

[4] G. Nielson and B. Hamann, *The asymptotic decider: resolving the ambiguity in Marching Cubes* In: Visualization, 29-38, 1991.

[5] G. Weber, O. Kreylos, T. Ligocki, J. Shalf, H. Hagen, and B. Hamann, *Extraction of crack-free isosurfaces from adaptative mesh refinement data* In: VisSym, 2001.

[6] R. Gallagher, *Computer visualization: graphics techniques for cientific and engineering analysis* : CRC Press, 1995.

[7] A. Guéziec and R. Hummel, *Exploting trianguled surface extraction using tetrahedral decomposition* IEEE Trans. on Visualiz. **1** (4), 328-342, 1995.

[8] G. Treece, R. Prager, and A. Gee, *Regularized marching tetrahedra: improved isosufarce extraction* Comp. Graphichs **23**, 583-598, 1999.

[9] B. Carneiro, C. Silva, and A Kaufman, *Tetra-Cubes: an algorithm to generate 3D isosurface based upon tetahedra* In: Sibgrapi, 205-210, 1996.

[10] B. Payne and A. Toga, *Surface mapping brain function on 3D models* IEEE Comp. Graph. and Applic. **10** (5), 33-41, 1990.

[11] D. C. Banks and S. Linton, *Counting cases in marching cubes: Toward a generic algorithm for producing substitopes* In: Visualization, 19-24, 2003.

[12] R. Gallagher, *Span filter: An optimization scheme for volume visualization of large finite element models* In: Visualization, 68-75, 1991.

[13] M. Giles and R. Haimes, *Advanced interactive visualization for CFD* Computing Syst. in Ingineering **1** (1), 51-62, 1990.

[14] H. Shen and C. Johnson, *Sweeping simplices: a fast isosurface extraction algorithm for unstructured grids* In: Visualization, 143-150, 1995.

[15] J. Wilhelms and A. van Gelder, *Octrees for faster isosurface generation* ACM Trans. on Graph. **11** (3), 201-227, 1992.

[16] Y. Livnat, H.W. Shen and C. Johnson, *A near optimal isosurface extraction algorithm using span space* IEEE Trans. on Visualiz. **2** (1), 73-84, 1996.

[17] J. Gao and H.W. Shen, *Hardware-assited view-dependent isosurface extraction using spherical partition* In: VisSym, 267-276, 2003.

[18] C. Bajaj, V. Pascucci and D. Schikore, *Fast isocontouring for improved interactivity* In: VolVis, 39-46, 1996.

[19] T. Itoh, Y. Yamaguchi and K. Koyamada, *Fast isosurface generation using the volume thinning algorithm* IEEE Trans. on Visualiz. **7** (1), 32-46, 2001.

[20] M. van Kreveld, R. van Oostrum and C. Bajaj, *Contour trees and small seed sets for isosurface traversal* In: SoCG, 212-219, 1997.

[21] T. Crockett, *An introduction to parallel rendering* Parallel Comp. **23**, 819-843, 1997.

[22] A. Clematis, D. d'Agostinod, W. de Marco and V. Gianuzzi *A web based isosurface extraction system for heterogeneous clients* In: Euromicro, 148-156, 2003.

[23] T.S. Newman and H. Yi, *A survey of the marching cubes algorithm* Comp. and Graph.**30**, 854-879, 2006.

[24] P. Bhaniramka, R. Wenger and R. Crawfis, *Isosurface construction in any dimension using convex hulls* IEEE Trans. on Visualiz. **10** (2), 130-141, 2004.

[25] J.C. Roberts and S. Hill, *Piecewise linear hypersurfaces using the marching cubes algorithm* In: Visual Data Exploration and Analysis, 170-181, 1999.

[26] Y. Matsumoto, *An introduction to Morse Theory* Translations of Mathematical Monographs, 208, 2002.

[27] *GAP: Groups, Algorithms, and Programming*