Least squares estimation of curvature and torsion

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Abstract. This work analyzes a new method for curvature and torsion estimation based on local parametric approximation. We show the convergence analysis and noise impact of the method.
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1 Introduction

Consider a collection of samples $\mathbf{P}$ of a planar or a spacial curve $\mathbf{r}$, i.e., a finite sequence of m *sample points* $\{\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_m\}$ of $\mathbf{r}$. We admit the presence of noise. In this report we describe and analyze a method for estimating the tangent line, the curvature and the torsion of the curve $\mathbf{r}$ at a point of $\mathbf{P}$.

The method is based on least squares parametric curve fitting. More precisely, we fit a parametric parabola to the data in the planar case and a parametric cubic in the spatial case. We prove the convergence of our curvature estimations under reasonable conditions over the sampling of the curve and the amplitude of the noise.

2 Model and notations

Consider a piecewise–linear approximation $\mathbf{P}$ of a smooth curve $\mathbf{r}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. We admit some noise in the samples. In this theoretical analysis, we will assume that the curve is parameterized by arc–length. For curvature estimation, we need to estimate the first and second derivatives of the coordinate functions $x(s)$,$y(s)$ and $z(s)$, while for torsion estimation we need to estimate the first, second and third derivatives of $x(s)$, $y(s)$ and $z(s)$.

Let us fix a sample point $\mathbf{p}_0$ in $\mathbf{P}$. We shall use a window of $2q + 1$ points of $\mathbf{P}$ around $\mathbf{p}_0$, so we write $\mathbf{P} = \{\mathbf{p}_{-q}, \mathbf{p}_{-q+1}, ..., \mathbf{p}_q\}$. Assuming that $\mathbf{p}_0 = \mathbf{r}(0)$ is the origin, we can write

$$\left\{\begin{array}{l} x(s) = x'(0)\, s + \frac{1}{2}\, x''(0)\, s^2 + g_1(s) s^3 \\ y(s) = y'(0)\, s + \frac{1}{2}\, y''(0)\, s^2 + g_2(s) s^3 \\ z(s) = z'(0)\, s + \frac{1}{2}\, z''(0)\, s^2 + g_3(s) s^3 \end{array}\right.$$

for the second order approximation and

$$\left\{\begin{array}{l} x(s) = x'(0)s + \frac{1}{2}x''(0)s^2 + \frac{1}{6}x'''(0)s^3 + g_1(s)s^4 \\ y(s) = y'(0)s + \frac{1}{2}y''(0)s^2 + \frac{1}{6}y'''(0)s^3 + g_2(s)s^4 \\ z(s) = z'(0)s + \frac{1}{2}z''(0)s^2 + \frac{1}{6}z'''(0)s^3 + g_3(s)s^4 \end{array}\right.$$

for the third order approximation, with $g_i(s) \to 0$ when $s \to 0$. In the case of planar curves, the equations corresponding to the z coordinate should be omitted. Since $\mathbf{p}_i$ are samples of the curve associated to the value of arc–length s_i, we can write

$$\left\{\begin{array}{l} x_i = x'(0)\, s_i + \frac{1}{2}\, x''(0)\, s_i^2 + g_1(s_i)\, s_i^3 + \eta_{x,i} \\ y_i = y'(0)\, s_i + \frac{1}{2}\, y''(0)\, s_i^2 + g_2(s_i)\, s_i^3 + \eta_{y,i} \\ z_i = z'(0)\, s_i + \frac{1}{2}\, z''(0)\, s_i^2 + g_3(s_i)\, s_i^3 + \eta_{z,i} \end{array}\right.$$

in the first case and

$$\left\{\begin{array}{l} x_i = x'(0)s_i + \frac{1}{2}x''(0)s_i^2 + \frac{1}{6}x'''(0)s_i^3 + g_1(s_i)s_i^4 + \eta_{x,i} \\ y_i = y'(0)s_i + \frac{1}{2}y''(0)s_i^2 + \frac{1}{6}y'''(0)s_i^3 + g_2(s_i)s_i^4 + \eta_{y,i} \\ z_i = z'(0)s_i + \frac{1}{2}z''(0)s_i^2 + \frac{1}{6}z'''(0)s_i^3 + g_3(s_i)s_i^4 + \eta_{z,i} \end{array}\right.$$

in the second case, where η_i is the noise corresponding to the point $\mathbf{p}_i$. We shall assume that the random variables η_i are independent and identically distributed (i.i.d.) with zero mean and variance σ^2. We aim to estimate, from the samples, $(\mathbf{r}'(0), \mathbf{r}''(0))$ for curvature estimation and $(\mathbf{r}'(0), \mathbf{r}''(0), \mathbf{r}'''(0))$ for curvature and torsion estimation. To obtain these estimates we shall follow a weighted least squares approach.

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3 The weighted least squares approach

First of all, we need an estimate for the arc–length s_i. Define Δl_k as the length of the vector $\mathbf{p}_k\mathbf{p}_{k+1}$, where k ranges from $-q$ to $(q-1)$. The arc–length estimator from $\mathbf{p}_0$ to $\mathbf{p}_i$ is defined as $l_i = \sum_{k=0}^{i-1} \Delta l_k$, when $i > 0$, and $l_i = -\sum_{k=i}^{-1} \Delta l_k$, when $i < 0$.

(a) Curvature Estimators

For curvature estimation, we shall look for a quadratic curve of the form

$$\begin{cases} x(s) = x'(0)s + \frac{1}{2}x''(0)s^2 \\ y(s) = y'(0)s + \frac{1}{2}y''(0)s^2 \\ z(s) = z'(0)s + \frac{1}{2}z''(0)s^2 \end{cases}$$

that better fits the data in the weighted least squares sense. Our approach is to minimize the square error of each coordinate independently. We shall look for x_0' and x_0'' that minimize

$$E_x(x_0', x_0'') = \sum_{i=-q}^{q} w_i \left(x_i - x_0' l_i - \tfrac{1}{2}x_0''\,(l_i)^2\right)^2 .$$

as our estimates of $x'(0)$ and $x''(0)$, with a similar approach for the other coordinates.

The real numbers w_i are the weight of the point $\mathbf{p}_i$. Such numbers are to be chosen positive, relatively large for small $|l_i|$ and relatively small for large $|l_i|$. We can consider weights of the form $w_i = \alpha\, \exp(-\beta i^2)/i^k$, for example. Another possible choice is simply $w_i = 1$.

The above problems have a well-known solution. Consider the matrix

$$A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$$

and the vectors $\mathbf{t}_x = \begin{bmatrix} t_x^1 \\ t_x^2 \end{bmatrix}$, $\mathbf{t}_y = \begin{bmatrix} t_y^1 \\ t_y^2 \end{bmatrix}$ and $\mathbf{t}_z = \begin{bmatrix} t_z^1 \\ t_z^2 \end{bmatrix}$, where

$$\begin{cases} a_1 = & & \sum_{i=-q}^{q} w_i\,(l_i)^2 \\ a_2 = & \frac{1}{2} & \sum_{i=-q}^{q} w_i\,(l_i)^3 \\ a_3 = & \frac{1}{4} & \sum_{i=-q}^{q} w_i\,(l_i)^4 \\ t_x^1 = & & \sum_{i=-q}^{q} w_i\,l_i\,(x_i) \\ t_x^2 = & \frac{1}{2} & \sum_{i=-q}^{q} w_i\,(l_i)^2\,(x_i) \\ t_y^1 = & & \sum_{i=-q}^{q} w_i\,l_i\,(y_i) \\ t_y^2 = & \frac{1}{2} & \sum_{i=-q}^{q} w_i\,(l_i)^2\,(y_i) \\ t_z^1 = & & \sum_{i=-q}^{q} w_i\,l_i\,(z_i) \\ t_z^2 = & \frac{1}{2} & \sum_{i=-q}^{q} w_i\,(l_i)^2\,(z_i) \end{cases} .$$

Then the vectors $\mathbf{x} = \begin{bmatrix} x_0' \\ x_0'' \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_0' \\ y_0'' \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} z_0' \\ z_0'' \end{bmatrix}$ are the unique solution of the equations $A \cdot \mathbf{x} = \mathbf{t}_x$, $A \cdot \mathbf{y} = \mathbf{t}_y$ and $A \cdot \mathbf{z} = \mathbf{t}_z$, respectively (see [1]).

Denoting $\mathbf{r}_0' = \begin{bmatrix} x_0' \\ y_0' \\ z_0' \end{bmatrix}$ and $\mathbf{r}_0'' = \begin{bmatrix} x_0'' \\ y_0'' \\ z_0'' \end{bmatrix}$, our curvature estimator is given by

$$\hat{\kappa}(\mathbf{p}_0) = \frac{\mathbf{r}_0' \times \mathbf{r}_0''}{||\mathbf{r}_0'||^3}$$

in the planar case and

$$\hat{\kappa}(\mathbf{p}_0) = \frac{||\mathbf{r}_0' \times \mathbf{r}_0''||}{||\mathbf{r}'||^3}$$

in the spatial case.

(b) Torsion Estimators

In the case of spatial curves, if we want to estimate the the torsion, we need to fit the data to cubic parametric curves. Assume that $\mathbf{p}_0 = \mathbf{r}(0) = (0,0,0)$. We shall fit our data to curves of the form

$$\begin{cases} x(s) = x_0'\, s + \frac{1}{2}\, x_0''\, s^2 + \frac{1}{6}\, x_0'''\, s^3 \\ y(s) = y_0'\, s + \frac{1}{2}\, y_0''\, s^2 + \frac{1}{6}\, y_0'''\, s^3 \\ z(s) = z_0'\, s + \frac{1}{2}\, z_0''\, s^2 + \frac{1}{6}\, z_0'''\, s^3 \end{cases} ,$$

where s a parameter.

As in the planar case, we shall look for x_0', x_0'' and x_0''' that minimize

$$E_x(x_0', x_0'', x_0''') \quad = \quad \sum_{i=-q}^{q} w_i \Big(x_i - \big(x_0' s_i + \tfrac{1}{2} x_0'' s_i^2 + \tfrac{1}{6} x_0''' s^3 \big) \Big)^2 . \tag{1}$$

A similar approach is used to estimate y_0', y_0'', y_0''', z_0', z_0'' and z_0'''.

Considering the same estimative l_i of the parameter s_i proposed above, define

$$A = \begin{bmatrix} a_1\ a_2\ a_4 \\ a_2\ a_3\ a_5 \\ a_4\ a_5\ a_6 \end{bmatrix}$$

and the vectors $\mathbf{t}_x = \begin{bmatrix} t_x^1 \\ t_x^2 \\ t_x^3 \end{bmatrix}$, $\mathbf{t}_y = \begin{bmatrix} t_y^1 \\ t_y^2 \\ t_y^3 \end{bmatrix}$ and $\mathbf{t}_z = \begin{bmatrix} t_z^1 \\ t_z^2 \\ t_z^3 \end{bmatrix}$, where

$$\begin{cases} a_4 = & \frac{1}{6} & \sum_{i=-q}^{q} w_i\, (l_i)^4 \\ a_5 = & \frac{1}{12} & \sum_{i=-q}^{q} w_i\, (l_i)^5 \\ a_6 = & \frac{1}{36} & \sum_{i=-q}^{q} w_i\, (l_i)^6 \\ t_x^3 = & \frac{1}{6} & \sum_{i=-q}^{q} w_i\, l_i^3\, (x_i) \\ t_y^3 = & \frac{1}{6} & \sum_{i=-q}^{q} w_i\, l_i^3\, (y_i) \\ t_z^3 = & \frac{1}{6} & \sum_{i=-q}^{q} w_i\, l_i^3\, (z_i) \end{cases} .$$

Then the vectors $\mathbf{x} = \begin{bmatrix} x_0' \\ x_0'' \\ x_0''' \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_0' \\ y_0'' \\ y_0''' \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} z_0' \\ z_0'' \\ z_0''' \end{bmatrix}$ are the unique solution of the equations $A \cdot \mathbf{x} = \mathbf{t}_x$, $A \cdot \mathbf{y} = \mathbf{t}_y$ and $A \cdot \mathbf{z} = \mathbf{t}_z$, respectively (see [1]).

Denoting $\mathbf{r}_0''' = \begin{bmatrix} x_0''' \\ y_0''' \\ z_0''' \end{bmatrix}$, our torsion estimator is given by

$$\hat{\tau}(\mathbf{p}_0) = -\frac{\left(\mathbf{r}_0' \times \mathbf{r}_0''\right) \cdot \mathbf{r}_0'''}{||\mathbf{r}_0' \times \mathbf{r}_0''||^2}$$

4 Convergence analysis of the curvature estimators

In the following, we shall denote:

$$\begin{aligned} \delta &= \max\left\{ |l_{-q}|\,, |l_q| \right\} \\ K_0 &= \max\{|\kappa(s)|\,, -\delta \le s \le \delta\}, \\ K_1 &= \max\{|\kappa'(s)|\,, -\delta \le s \le \delta\}, \\ T_0 &= \max\{|\tau(s)|\,, -\delta \le s \le \delta\}, \end{aligned}$$

where $\kappa(s)$ is the curvature and $\tau(s)$ the torsion of $\mathbf{r}$ at s.

(a) Sampled curve without noise

There are some assumptions that we must consider in order to expect a good behavior of the curvature estimates. The first one is that the product $K_0\delta$ should be small, which corresponds to dense sampling in high curvature regions. If not, some samples are too far from $\mathbf{p}_0$ to be correctly used in the estimate of the first derivatives of $\mathbf{r}$ at $\mathbf{p}_0$. The second is that $K_1\delta$ and $T_0\delta$ should be small, which corresponds to dense sampling in regions where curvature is changing rapidly. If this does not occur, some samples must be considered too far from the basic point to help in the second derivatives estimates.

We shall now make precise these statements. Let

$$\phi_1 = \frac{1}{\det(A)}\frac{l_i}{4}\left(\sum_{i=-q}^{q} w_i l_i \sum_{i=-q}^{q} w_i l_i^4 - \sum_{i=-q}^{q} w_i l_i^2 \sum_{i=-q}^{q} w_i l_i^3\right)$$

and

$$\phi_2 = \frac{1}{\det(A)}\frac{l_i^2}{4}\left(\sum_{i=-q}^{q} w_i l_i^2 \sum_{i=-q}^{q} w_i l_i^2 - \sum_{i=-q}^{q} w_i l_i \sum_{i=-q}^{q} w_i l_i^3\right)$$

Proposition 1 (Convergence without noise) *(a) Assume that $K_0\delta \le \varepsilon$. Then*

$$|x_0' - x'(0)| \le \phi_1\left(d_1(\varepsilon) + \left|\frac{x''(0)\delta}{2}\right|\right),$$

where $d_1(\varepsilon)$ is defined in lemma 3 below.
(b) Assume that $K_0\delta \le \varepsilon$, $K_1\delta \le \varepsilon$ and $T_0\delta \le \varepsilon$. Then

$$|x_0'' - x''(0)| \le \phi_2 d_2(\varepsilon),$$

where $d_2(\varepsilon)$ is defined in lemma 4 below.

We observe that ϕ_1 and ϕ_2 are homogeneous of degree zero, i.e., they do not change if we multiply all l_i by some constant. The above proposition says that the curvature estimation is convergent in the sense that if we reduce sufficiently the value of δ without changing the proportions of the l_i, the difference between the real and estimated will be arbitrarily small.

In order to prove this proposition, we need some lemmas:

Lemma 2 *Assume that $\delta K_0 < \varepsilon$. Then $sinc(\varepsilon/2)|s_i| \le |l_i| \le |s_i|$, where $sinc(\varepsilon) = \frac{\sin(\varepsilon)}{\varepsilon}$.*

Proof: For any $i > 0$, we have that $s_i - l_i = \sum_{k=0}^{i-1} \Delta e_k$, where Δe_k is the difference between the arc–length of the curve between $\mathbf{p}_k$ and $\mathbf{p}_{k+1}$ and the corresponding linear segment. Since the curvature is bounded by K_0, the difference Δe_k is maximum when the curve is a circle of radius $\dfrac{1}{K_0}$. In this case, the corresponding central angle is $K_0\Delta s_k$ and so

$$\Delta l_k \ge \frac{2\sin\left(\frac{K_0\Delta s_k}{2}\right)}{K_0}.$$

We conclude that

$$l_i = \sum_{k=1}^{i}\Delta l_k \ge \sum_{k=1}^{i}\frac{2}{K_0\Delta s_k}\sin\left(\frac{K_0\Delta s_k}{2}\right)\Delta s_k \ge sinc(\varepsilon/2)s_i,$$

thus proving the lemma. ■

Lemma 3 *Assume that $\delta K_0 < \varepsilon$. Then*

$$|x_i - x'(0)l_i| \le d_1(\varepsilon)|l_i| \quad ,$$

where $d_1(\varepsilon) = \dfrac{x'(0)(1 - sinc(\varepsilon/2))}{sinc(\varepsilon/2)} + \dfrac{\varepsilon}{sinc(\varepsilon/2)}$.

Proof: Since

$$|x''(s)|\delta \leq K_0\delta \leq \varepsilon$$

for $s \in (-\delta, \delta)$, we obtain

$$|x'(s) - x'(0)| \leq \varepsilon \quad .$$

Hence, for any i

$$x'(0) - \varepsilon \leq \frac{x_i}{s_i} \leq x'(0) + \varepsilon \quad .$$

We can assume without loss of generality that $x'(0) > 0$. using lemma 2,

$$(x'(0) - \varepsilon)l_i \leq x_i \leq (x'(0) + \varepsilon)\frac{l_i}{sinc(\varepsilon/2)} \quad .$$

We can conclude then that

$$|x_i - x'(0)l_i| \leq d_1(\varepsilon)|l_i| \quad .$$

where $d_1(\varepsilon) = \dfrac{x'(0)(1 - sinc(\varepsilon/2))}{sinc(\varepsilon/2)} + \dfrac{\varepsilon}{sinc(\varepsilon/2)}$. ■

Lemma 4 *Assume that $\delta K_0 < \varepsilon$, $\delta K_1 < \varepsilon$ and $\delta T_0 < \varepsilon$. Then*

$$\left|x_i - x'(0)l_i - \frac{x''(0)(l_i)^2}{2}\right| \leq d_2(\varepsilon)\frac{(l_i)^2}{2} \quad ,$$

where

$$d_2(\varepsilon) = x''(0)\frac{1 - sinc(\varepsilon/2)^2}{sinc(\varepsilon/2)^2} + \frac{\varepsilon}{sinc(\varepsilon/2)^2}(1 + \frac{x'(0)K_0}{6}).$$

Proof: Using that $K_1\delta \leq \varepsilon$ and $\delta T_0 < \varepsilon$ we obtain that $|x''(s) - x''(0)| \leq \varepsilon$, for any $s \in (-\delta, \delta)$.
Then, for any i,

$$(x''(0) - \varepsilon)\frac{s_i^2}{2} \leq x_i - x'(0)s_i \leq \frac{s_i^2}{2}(x''(0) + \varepsilon) \quad .$$

By lemma 2

$$(x''(0) - \varepsilon)\frac{(l_i)^2}{2} \leq x_i - x'(0)s_i + x'(0)l_i - x'(0)l_i \leq \frac{(l_i)^2}{2sinc(\varepsilon/2)^2}(x''(0) + \varepsilon) \quad .$$

It follows that

$$\frac{x''(0)(l_i)^2}{2} - \frac{\varepsilon(l_i)^2}{2} \leq x_i - x'(0)l_i \leq \frac{x''(0)(l_i)^2}{2sinc(\varepsilon/2)^2} + \frac{\varepsilon(l_i)^2}{2sinc(\varepsilon/2)^2} + x'(0)(s_i - l_i)$$

and so

$$-\frac{\varepsilon(l_i)^2}{2} \leq x_i - x'(0)l_i - \frac{x''(0)(l_i)^2}{2} \leq \frac{x''(0)(l_i)^2}{2sinc(\varepsilon/2)^2} + \frac{\varepsilon(l_i)^2}{2sinc(\varepsilon/2)^2} - \frac{x''(0)(l_i)^2}{2} + x'(0)s_i(1 - \frac{l_i}{s_i}) \quad .$$

By lemma 2

$$1 - \frac{l_i}{s_i} \leq 1 - \frac{\sin\left(\frac{K_0 s_i}{2}\right)}{\frac{K_0 s_i}{2}} \quad .$$

Since $1 - \frac{\sin(v)}{v} \leq \frac{v^2}{3}$, we conclude that

$$x'(0)s_i(1 - \tfrac{l_i}{s_i}) \leq \tfrac{x'(0)s_i K_0^2 s_i^2}{12} \leq \tfrac{x'(0)K_0\varepsilon s_i^2}{12} \leq \tfrac{x'(0)K_0\varepsilon}{6}\tfrac{(l_i)^2}{2sinc(\varepsilon/2)^2} \quad .$$

Returning to the main calculations we have

$$\left|x_i - x'(0)l_i - \frac{x''(0)(l_i)^2}{2}\right| \leq d_2(\varepsilon)\frac{(l_i)^2}{2} \quad ,$$

where

$$d_2(\varepsilon) = x''(0)\frac{1 - sinc(\varepsilon/2)^2}{sinc(\varepsilon/2)^2} + \frac{\varepsilon}{sinc(\varepsilon/2)^2}(1 + \frac{x'(0)K_0}{6}),$$

thus proving the lemma. ■

We can now prove the main proposition. Simple matrix calculations show that

$$\begin{bmatrix} x'_0 - x'(0) \\ x''_0 - x''(0) \end{bmatrix} = \frac{x_i - x'_0 l_i - \frac{1}{2}x''_0 l_i^2}{\det(A)} \begin{bmatrix} \frac{1}{4}(\sum_{i=-q}^q w_i l_i \sum_{i=-q}^q w_i l_i^4 - \sum_{i=-q}^q w_i l_i^2 \sum_{i=-q}^q w_i l_i^3) \\ \frac{1}{2}(\sum_{i=-q}^q w_i l_i^2 \sum_{i=-q}^q w_i l_i^2 - \sum_{i=-q}^q w_i l_i \sum_{i=-q}^q w_i l_i^3) \end{bmatrix}$$

Now the proposition follows easily from lemmas 3 and 4.

(b) Sampled curve with noise

We are assuming that the noise at each sample is a random vector η_i, independent for each sample, with mean 0 and standard deviation σ. In order to have a good estimate of the derivatives, we must assume that the noise is not too big relatively to the distance between samples. In this section, we shall make this statement more precise.

Denote by $\delta_1 = \min(|l_1|, |l_{-1}|)$. If we want to use the above estimations in the noisy case, the ratios $\frac{\sigma}{\delta_1}$ and $\frac{\sigma}{\delta_1}^2$ should be small. If not, the noise is too strong for us to guarantee the estimation for $(x'(0), y'(0))$ and $(x''(0), y''(0))$, respectively. This is the content of the following proposition:

Proposition 5 (Convergence with noise) *(a) Assume that $\sigma \leq \gamma\delta_1$. Then the error of estimation $|x'_0 - x'(0)|$ is bounded by the sum of the errors of proposition 1(a) and a random variable of zero mean and variance less than $\psi_1\gamma$, where*

$$\psi_1 = \frac{\delta_1}{4\det(A)}\left(\sum_{i=-q}^q w_i^2 l_i^2 \left(\sum_{i=-q}^q w_i l_i^4\right)^2 + \sum_{i=-q}^q w_i^2 l_i^4 \left(\sum_{i=-q}^q w_i l_i^3\right)^2\right)^{1/2}$$

.

(b) Assume that $\sigma \leq \gamma{\delta_1}^2$. Then the error of estimation $|x''_0 - x''(0)|$ is bounded by the sum of the errors of proposition 1(b) and a random variable of zero mean and variance less than $\psi_2\gamma$, where

$$\psi_2 = \frac{\delta_1^2}{2\det(A)}\left(\sum_{i=-q}^q w_i^2 l_i^4 \left(\sum_{i=-q}^q w_i l_i^2\right)^2 + \sum_{i=-q}^q w_i^2 l_i^2 \left(\sum_{i=-q}^q w_i l_i^3\right)^2\right)^{1/2}$$

.

We observe that ψ_1 and ψ_2 are homogeneous of degree zero, i.e., they do not change if we multiply all l_i by some constant. The above proposition says that the curvature estimation is convergent in the sense that if we reduce sufficiently the noise standard deviation σ without changing the proportions of the l_i, the difference between the real and estimated will be arbitrarily small.

In the particular case where the samples are symmetrically distributed around $\mathbf{p}_0$ and the weights w_i are equal to 1, we have $\psi_1 = \frac{\delta_1}{4(\sum_{i=-q}^q l_i^2)^{1/2}}$ and $\psi_2 = \frac{\delta_1^2}{2(\sum_{i=-q}^q l_i^4)^{1/2}}$. If q is big, $\psi_1 = O(q^{-3/2})$ and $\psi_2 = O(q^{-5/2})$.

Proof: We have to analyze the effect of noise in the calculations of proposition 1 . If we change x_i by $x_i + \eta_{x,i}$ in the formulas for x'_0 and x''_0, the estimates will change by the random variables e'_x and e''_x satisfying the equation

$$\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \begin{bmatrix} e'_x \\ e''_x \end{bmatrix} = \begin{bmatrix} \sum_{i=-q}^q w_i l_i \eta_{x,i} \\ \frac{1}{2}\sum_{i=-q}^q w_i l_i^2 \eta_{x,i} \end{bmatrix}$$

Hence the errors have zero mean and standard deviation given by

$$\begin{cases} \mathrm{std}(e'_x) & = & \frac{\sigma}{4\det(A)}\left(\sum_{i=-q}^q w_i^2 l_i^2 \left(\sum_{i=-q}^q w_i l_i^4\right)^2 + \sum_{i=-q}^q w_i^2 l_i^4 \left(\sum_{i=-q}^q w_i l_i^3\right)^2\right)^{1/2} \\ \mathrm{std}(e''_x) & = & \frac{\sigma}{2\det(A)}\left(\sum_{i=-q}^q w_i^2 l_i^4 \left(\sum_{i=-q}^q w_i l_i^2\right)^2 + \sum_{i=-q}^q w_i^2 l_i^2 \left(\sum_{i=-q}^q w_i l_i^3\right)^2\right)^{1/2} \end{cases}$$

thus proving the proposition. ■

References

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