Convergence of affine estimators on parabolic polygons

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Abstract. This technical report proves the convergence of affine estimators on parabolic polygons.


Figure 1: Affine length estimations on a lemniscate with 51 samples, before and after an affine transformation.

1 Convergence of the affine estimators

Let \( x(s) = (x(s), y(s)), -u \leq s \leq t \), be a convex plane curve parameterized by affine arc-length, with affine curvature \( \mu(s) \). Denote \( \mu = \mu(0), \nu = \mu'(0) \). Assume that \( x(0) = (0, 0), x'(0) = (1, 0) \) and \( \nu'(0) = (0, 1) \). We can write
\[
x(s) = s - \frac{\nu'}{\nu} \cdot s^3 - \frac{\mu'}{\nu^2} \cdot s^4 + O(s^5) \quad \text{and} \quad y(s) = \frac{s^2}{2} + \frac{\nu}{\nu^2} \cdot s^4 - \frac{\mu}{\nu^3} \cdot s^5 + O(s^6).
\]

Consider that the samples are \( x_i = x(0), x_{i-1} = x(-u) \) and \( x_{i+1} = x(t) \). Denoting by \( z(t) = (z(t), 0) \) the intersection of the lines defined by \( x'(0) \) and \( x'(t) \), the affine length of the parabola is \( L(t) = \frac{\sqrt{2}}{4z(t)y(t)} \). Let
\[
T = \begin{bmatrix}
A(t) & B(t) \\
0 & D(t)
\end{bmatrix}
\]

be the affine transformation that fixes \( (x(0), x'(0)) \) and takes \( (x_i, x'_{i+1}) \) to \( ((t, \frac{x^2}{2}), (1, t)) \).

Then direct calculations show that:
\[
\begin{align*}
z(t) &= \frac{1}{2} \cdot t + \frac{\mu}{\nu^2} \cdot t^3 + \frac{\nu}{\nu^3} \cdot t^4 + O(t^5), \\
L(t) &= t + \frac{\mu}{\nu^2} \cdot t^3 + O(t^4), \\
A(t) &= 1 + \frac{\mu}{\nu^2} \cdot t^2 + \frac{\nu}{\nu^3} \cdot t^3 + O(t^3), \\
B(t) &= -\frac{\nu}{\nu^3} \cdot t^2 - \frac{\mu}{\nu^2} \cdot t^3 + O(t^4), \\
D(t) &= 1 - \frac{\nu}{\nu^3} \cdot t^2 - \frac{\mu}{\nu^2} \cdot t^3 + O(t^4).
\end{align*}
\]

The affine curvature estimator is given by
\[
\pi_i(t, u) = -2 \frac{|q''(t), q''(u)|}{L(t) + L(u)},
\]
where \( q'' = (B, D) \). Direct calculi show that \( \pi_i(t, u) = \mu_i + \frac{1}{10}(t - u) \nu + O(t^2 + u^2) \), which proves the convergence of the affine curvature estimator.

Assuming that the ratios between the affine lengths \( t, u, \nu \) are bounded, we obtain
\[
\overline{\pi}_{i+1} - \overline{\pi}_i = \nu_i[u + \frac{3}{10}(t - u) - \frac{3}{10}(u - v)] + O(t^2 + u^2 + v^2),
\]
and so \( \pi_i = \nu_i + O(t + u + v) \), thus proving the convergence of the derivative of the affine curvature estimator.

Preprint MAT. 12/07, communicated on August 3rd, 2007 to the Department of Mathematics, Pontificia Universidade Católica — Rio de Janeiro, Brazil.