Convergence of affine estimators on parabolic polygons

THOMAS LEWINER AND MARCOS CRAIZER

Departament of Matematics — Pontifícia Universidade Católica — Rio de Janeiro — Brazil
www.mat.puc--rio.br/~{tomlew,craizer}.

Abstract. This technical report proves the convergence of affine estimators on parabolic polygons.

Keywords: ***Affine Differential Geometry. Affine Curvature. Affine Lenght. Discrete estimators.***

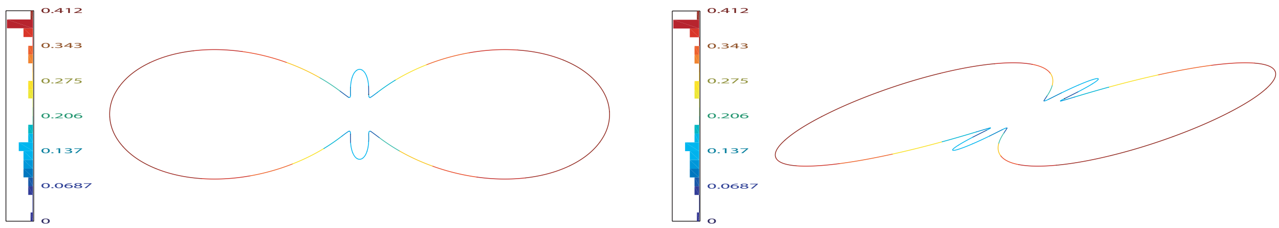

Figure 1: *Affine length estimations on a lemniscate with 51 samples, before and after an affine transformation.*

1 Convergence of the affine estimators

Let $\mathbf{x}(s) = (x(s), y(s))$, $-u \le s \le t$, be a convex plane curve parameterized by affine arc-length, with affine curvature $\mu(s)$. Denote $\mu = \mu(0), \nu = \mu'(0)$. Assume that $\mathbf{x}(0) = (0,0)$, $\mathbf{x}'(0) = (1,0)$ and $\mathbf{x}''(0) = (0,1)$. We can write $x(s) = s - \frac{\mu}{6} \cdot s^3 - \frac{\nu}{24} \cdot s^4 + O(s^5)$ and $y(s) = \frac{s^2}{2} + \frac{\mu}{24} \cdot s^4 + -\frac{\nu}{60} \cdot s^5 + O(s^6)$.

Consider that the samples are $\mathbf{x}_i = \mathbf{x}(0)$, $\mathbf{x}_{i-1} = \mathbf{x}(-u)$ and $\mathbf{x}_{i+1} = \mathbf{x}(t)$. Denoting by $\mathbf{z}(t) = (z(t), 0)$ the intersection of the lines defined by $\mathbf{x}'(0)$ and $\mathbf{x}'(t)$, the affine length of the parabola is $L(t) = \sqrt[3]{4z(t)y(t)}$. Let

$$T = \begin{bmatrix} A(t) & B(t) \\ 0 & D(t) \end{bmatrix}$$

be the affine transformation that fixes $(\mathbf{x}(0), \mathbf{x}'(0))$ and takes $(\mathbf{x}_{i+1}, \mathbf{x}'_{i+1})$ to $((t, \frac{t^2}{2}), (1, t))$.

Then direct calculations shows that :

$$\left\{ \begin{array}{rclllll} z(t) & = & \frac{1}{2} \cdot t + & \frac{\mu}{24} \cdot t^3 + & \frac{\nu}{60} \cdot t^4 + & O(t^5), \\ L(t) & = & t + & & & O(t^5), \\ A(t) & = & 1 + & \frac{\mu}{12} \cdot t^2 + & \frac{\nu}{30} \cdot t^3 + & O(t^4), \\ B(t) & = & -\frac{\mu}{2} \cdot t - & \frac{3\nu}{20} \cdot t^2 + & & O(t^3), \\ D(t) & = & 1 - & \frac{\mu}{12} \cdot t^2 - & \frac{\nu}{30} \cdot t^3 + & O(t^4). \end{array} \right.$$

The affine curvature estimator is given by

$$\overline{\mu}_i(t, u) = -2 \frac{|\mathbf{q}''(t), \mathbf{q}''(-u)|}{L(t) + L(u)}.$$

where $\mathbf{q}'' = (B, D)$. Direct calculii show that $\overline{\mu}_i(t, u) = \mu_i + \frac{3}{10}(t - u)\nu + O(t^2 + u^2)$, which proves the convergence of the affine curvature estimator.

Assuming that the ratios between the affine lengths t, u, v are bounded, we obtain

$$\overline{\mu}_{i+1} - \overline{\mu}_i = \nu_i [u + \frac{3}{10}(t - u) - \frac{3}{10}(u - v)] + O(t^2 + u^2 + v^2),$$

and so $\overline{\nu}_i = \nu_i + O(t + u + v)$, thus proving the convergence of the derivative of the affine curvature estimator.

Preprint MAT. 12/07, communicated on August 3^{rd}, 2007 to the Department of Mathematics, Pontifícia Universidade Católica — Rio de Janeiro, Brazil.